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Prepared by C. R. ORTLOFF
Aerodynamics and Propulsion Research Laboratory

Prepared for COMMANDER SPACE SYSTEMS DIVISION
UNITED STATES AIR FORCE
Inglewood, California

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AN INVERSE METHOD FOR THE DETERMINATION OF THE THREE-DIMENSIONAL SUBSONIC FLOW REGION ABOUT BLUNTED BODIES WITHOUT AXIAL SYMMETRY AT SUPERSONIC FREE STREAM MACH NUMBERS

~~Prepared~~ by
C. R. Ortloff
Aerodynamics and Propulsion Research Laboratory

AEROSPACE CORPORATION
El Segundo, California

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ABSTRACT

A three-dimensional blunt body inverse technique is formulated in terms of two-component stream functions in the spirit of the Ferri, Vaglio-Laurin axisymmetric blunt body method. In a similar manner to the axisymmetric case, transformations are found that reduce the elliptic asymmetric shock layer to a transformed space in which shock, streamlines (or the intersections of stream surfaces), body, and sonic surface are a priori known, so that numerical analysis procedures are considerably simplified. Provided three equations in the three velocity components of the transformed space are numerically solved, then inverse transformations are given so that the corresponding velocities, pressure, and density at a point in the physical space may be determined. The class of given shocks considered are portions of prolate ellipsoids in the physical space.

CONTENTS

FIGURES

LIST OF SYMBOLS

$\alpha, \beta, \alpha', \beta'$	constants defined by Eqs. (12) and (12A)
γ	$= c_p/c_v$
Λ	boundary of the region R_1
ξ_1, ξ_2, ξ_3	prolate spheroidal curvilinear coordinates (Fig. 1)
ξ, η, ζ	arguments of the prolate spheroidal curvilinear coordinates
ρ	density
σ, τ, π	coordinates of the transformed space $\widetilde{R}$ (Fig. 2)
ψ, ϕ	three-dimensional stream functions
B^i_{jk}, C^i_{jk}	elements of the matrices defined by Eq. (87)
C	radius of the disc $\xi = 0$ on the $z = 0$ plane
$\hat{e}_1, \hat{e}_2, \hat{e}_3$	curvilinear unit vectors defined in Section I (Fig. 1)
f^n	contravariant velocity components
h_1, h_2, h_3	metrics of the ξ_1, ξ_2, ξ_3 curvilinear system
$\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$	metrics of the σ, τ, π coordinate frame
I_1, I_2	integrals of the $p\rho^{-\gamma}$ general integral, Eqs. (81) and (82)
J, j	Jacobian determinants
M	Mach number
p	pressure

LIST OF SYMBOLS (Continued)

$\tilde{R}$	the open region composed of all points within the (σ, τ, π) transformed shock layer
R_1	the open region composed of all points within the (ξ_1, ξ_2, ξ_3) shock layer
R	$= c_p - c_v$
S	entropy
$\tilde{u}, \tilde{v}, \tilde{w}$	velocity components of the transformed (σ, τ, π) coordinate frame (Fig. 2)
u, v, w	velocity components defined in Section I (Fig. 1)
$\overline{V}$	velocity vector at a point P in the $P(\xi_1, \xi_2, \xi_3) \epsilon R_1$ reference frame

<u>Subscripts</u>

s	denotes conditions on the shock front ξ_s

I. INTRODUCTION

In order to predict the subsonic inviscid flow field region about blunted bodies without axial symmetry, or axisymmetric or asymmetric bodies at an angle of attack, a detailed analysis of an improperly set elliptic system of nonlinear equations must be solved by a stable finite difference scheme. For axisymmetric bodies (at zero and at small angles of attack) many problems associated with a straightforward approach to the above problem have been minimized by transformation to a new space upon whose boundaries the boundary conditions are known (Refs. 1 and 2), and where velocity distributions at the axis and in the neighborhood of the sonic line are *a priori* determined and used as boundary data. In addition, the domain of influence of the sonic line is known *a priori*, leading to uniqueness of the computed body for a given analytic shock; further reasons for attempting an extension of this method for three-dimensional flows are obvious from the discussion presented in Ref. 2.

Previous analysis of regions of this type (Ref. 3) by series expansion methods, with a Clebsch transformation, generally leads to a great deal of computation, indicating the necessity of an analog to the comparatively simple and theoretically sound method of Refs. 1 and 2. In most cases of Ref. 3, two or three terms of the expansion (involving the solution of the associated ordinary nonlinear coupled systems of equations) were necessary for shock and sonic line shapes.

At present, numerical methods of the kind necessary to fully understand error analysis for the type of calculations suggested by the inverse problem have not been fully developed;[1] however, the methods of Refs. 4 and 5 suggest approaches for improperly set Laplace or Cauchy-Riemann systems with regard to round-off error analysis, approximation error, necessary step size, etc., for the linear problem.

Provided the sonic surface is known (or the elliptic solution can be extended sufficiently past the sonic surface as is sometimes done), a number of three-dimensional characteristics methods can be used[2] (Refs. 10, 11, and 12) to continue the solution into the hyperbolic domain.

In accordance with this discussion, then, an inverse method is developed in the spirit of the Ferri, Vaglio-Laurin method for three-dimensional flows about nonaxisymmetric zero angle of attack bodies. A transformation of the region R_1 (Fig. 1) to a region $\tilde{R}$ (Fig. 2) is effected, and subsequent development of a suitable stream function pair leads to a system of three nonlinear equations in the unknown velocity components in $\tilde{R}$, which are solvable by the method of Refs. 1 and 2, when extended to three-dimensional nets. Solutions in the transformed space are then transformable into the physical space for the actual flow and resulting streamlines.

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[1] Reflection methods for elliptic systems have been used to obtain symmetric hyperbolic transformed systems for the flow about blunted bodies at a small angle of attack (Ref. 4). For such systems, stable numerical techniques and error estimates are to be found (for example Ref. 5 through 9) in contrast to the "marching" method for improperly set elliptic equations.

[2] Many additional comments concerning the numerical and theoretical methods of Ref. 8 are presented in Vol. II of the report (in preparation).

In that the axis flow is not completely a priori determined from the resulting system of equations, it is recommended that the bounding surfaces be taken as the shock, sonic surface, and body approximation surface in the $\tilde{R}$ space for purposes of computation. From a computational standpoint, round-off errors, say on the initial surface, propagate exponentially and, therefore, obviate methods of filtering high frequencies for stable solutions; alternatively, for a high accuracy difference scheme, only a few steps are necessary across the shock layer so that growth of round-off error is lessened. For known flow boundaries, the latter alternative leads to difference equations always satisfying the sonic surface given boundary condition. From Ref. 2, from the fact that stream surfaces are a priori known, a lower-order difference scheme is necessary, since the order of the system has been reduced. Although details of the numerical analysis will not be dealt with in detail here, it is possible to see that many of the numerical simplifications of Refs. 1 and 2 carry over to the method to be indicated. Although solutions may be computed by a variety of methods extended from their axisymmetric counterpart, the method of Refs. 1 and 2 appears to be singular in the use of known information on the axis and sonic surface, leading to a more theoretically sound and numerically simple method in the extension to a higher dimension.

For a specified asymmetric portion of the shock surface given over the subsonic and transonic regions (here taken as a portion of a prolate ellipsoid), the method indicated in this report may be applied to determine the three-dimensional shock layer in detail and also the body shape.

Knowledge of the aerodynamic characteristics (pressures, velocities, etc.) in the nose region of a particular "lifting" surface may be determined by this method, and inputs to a downstream characteristics-routine established. In this manner, total aerodynamic analysis of fully three-dimensional lifting bodies may be theoretically performed in order to obtain design criteria for such bodies.

II. SOME PRELIMINARY CONSIDERATIONS FOR A THREE-DIMENSIONAL INVISCID BLUNT BODY INVERSE METHOD

The equations governing the steady three-dimensional rotational flow of a compressible inviscid gas may be written in general vector form as

$$\nabla \cdot \rho \overline{V}^* = 0 \tag{1}$$

$$\rho^*(\overline{V}^* \cdot \nabla)\overline{V}^* + \nabla p^* = 0 \tag{2}$$

$$\overline{V}^* \cdot \nabla(p^*/\rho^{*\gamma}) = 0 \tag{3}$$

where the $(^*)$ denotes dimensional quantities.

For a three-dimensional blunt body inverse technique formulated in terms of a curvilinear coordinate system, a coordinate surface corresponding to the shock surface is chosen as a portion of an ellipse of rotation about the z-axis. This portion is to lie in the half-space $x \leq 0$, where the $x = 0$ plane extends past the sonic point on the shock surface. The free stream velocity vector $\overline{V}_\infty$ is parallel to the x-axis. In oblate spheroidal coordinates the metrics are

$$h_1^2 = \frac{\cosh^2 \xi}{(\cosh^2 \xi - \cos^2 \eta)} \quad h_2^2 = \frac{\cos^2 \eta}{C^2(\cosh^2 \xi - \cos^2 \eta)}$$

$$h_3^2 = \frac{\cos^2 \zeta}{C^2 \cosh^2 \xi \cos^2 \eta} \tag{4}$$

for

$$\xi_1 = C \sinh \xi \quad \xi_2 = \sin \eta \quad \xi_3 = \sin \zeta \tag{4a}$$

The coordinates ξ_1, ξ_2, ξ_3 are the curvilinear coordinates; values of ξ, η, ζ are essentially parameters constant on families of the ξ_1, ξ_2, ξ_3 coordinate surfaces, respectively. Nondimensionalization by free-stream values,

$$u = \frac{u^*}{V_\infty}\ , \quad v = \frac{v^*}{V_\infty}\ , \quad w = \frac{w^*}{V_\infty}\ , \quad p = \frac{p^*}{\rho_\infty V_\infty^2}\ , \quad \rho = \frac{\rho^*}{\rho_\infty}\ ,$$

$$S = \frac{S^*}{R}(\gamma - 1) \tag{4b}$$

is next performed on the system (Eqs. 1 through 3). Note that the metrics (Eq. 4) are

$$h_1^2 = \frac{\xi_1^2 + C^2}{\xi_1^2 + C^2\xi_2^2} \quad h_2^2 = \frac{1 - \xi_2^2}{\xi_1^2 + C^2\xi_2^2} \quad h_3^2 = \frac{1 - \xi_3^2}{\left(\xi_1^2 + C^2\right)\left(1 - \xi_2^2\right)} \tag{4c}$$

and the surface ξ = const. corresponds to an ellipse of rotation about the z-axis; the surface η = const. corresponds to a one-sheet hyperbola of revolution about the z-axis; and ζ = const. corresponds to a plane through the z-axis (Fig. 1). The constant C corresponds to the origin-to-focus distance of the coordinate system, or the radius of the $\xi = 0$ disc.[3]

[3] C is the common focus point of the ξ = const., η = const. surfaces; the plane z = 0 minus the $\xi = 0$ disc is the surface $\eta = 0$.

For the range $0 < \xi \leq \xi_s$ (where ξ_s is taken to represent a portion of the prescribed shock in the half-space $x \leq 0$) $0 < \eta < \pi/2$, $0 < \zeta < \pi/2$, points (ξ, η, ζ) of the prolate-spheroidal coordinate system cover the space of the three-dimensional shock layer (R_1) lying between ξ_s and $B(\xi, \eta, \zeta)$, the coordinates of a computed body surface. The shock ξ_s and M_∞ are chosen such that the focus of ξ_s lies inside of the computed body corresponding to ξ_s. In this coordinate system (Fig. 1), then, Eqs. (1), (2), and (3) become, in scalar form

$$\frac{\partial}{\partial \xi_1}\left(h_1 \rho u \frac{h_2 h_3}{h_1}\right) + \frac{\partial}{\partial \xi_2}\left(h_2 \rho v \frac{h_1 h_3}{h_2}\right) + \frac{\partial}{\partial \xi_3}\left(h_3 \rho w \frac{h_1 h_2}{h_3}\right) = 0 \tag{5}$$

$$\frac{u}{h_1}\frac{\partial}{\partial \xi_1}(p\rho^{-\gamma}) + \frac{v}{h_2}\frac{\partial}{\partial \xi_2}(p\rho^{-\gamma}) + \frac{w}{h_3}\frac{\partial}{\partial \xi_3}(p\rho^{-\gamma}) = 0 \tag{6}$$

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial u}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial u}{\partial \xi_3} + \frac{v}{h_1 h_2}\left(u\frac{\partial h_1}{\partial \xi_2} - v\frac{\partial h_2}{\partial \xi_1}\right) + \frac{w}{h_1 h_3}\left(u\frac{\partial h_1}{\partial \xi_3} - w\frac{\partial h_3}{\partial \xi_1}\right) + \frac{1}{\rho h_1}\frac{\partial p}{\partial \xi_1} = 0 \tag{7}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial v}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial v}{\partial \xi_3} + \frac{w}{h_2 h_3}\left(v\frac{\partial h_2}{\partial \xi_3} - w\frac{\partial h_3}{\partial \xi_2}\right) + \frac{u}{h_2 h_1}\left(v\frac{\partial h_2}{\partial \xi_1} - u\frac{\partial h_1}{\partial \xi_2}\right) + \frac{1}{\rho h_2}\frac{\partial p}{\partial \xi_2} = 0 \tag{8}$$

$$\frac{u}{h_1}\frac{\partial w}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial w}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial w}{\partial \xi_3} + \frac{u}{h_3 h_1}\left(w\frac{\partial h_3}{\partial \xi_3} - u\frac{\partial h_1}{\partial \xi_3}\right) + \frac{v}{h_3 h_2}\left(w\frac{\partial h_3}{\partial \xi_2} - v\frac{\partial h_2}{\partial \xi_3}\right) + \frac{1}{\rho h_3}\frac{\partial p}{\partial \xi_3} = 0 \tag{9}$$

$$\left[\begin{array}{l} \text{or } v^j\left(\frac{\partial v^i}{\partial x^j} + \left\{ {i \atop \ell\ j} \right\} v^\ell\right) + \frac{1}{\rho}\, g^{ij}\, \frac{\partial p}{\partial x^j} = 0 \ , \quad (\rho v^i)_{,i} = 0 \quad , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v^j\, \frac{\partial (p\rho^{-\gamma})}{\partial x^j} = 0 \\ \text{where } v^i \text{ are the contravariant velocity components.} \end{array}\right] \quad \text{(5a)-(9a)}$$

A point $P(\xi_1, \xi_2, \xi_3) \in (R_1 \cap \Lambda)$ is formed by the intersection of the orthogonal (ξ_1, ξ_2, ξ_3) surfaces; at P, on the plane ζ = const., unit vectors $\hat{e}_1$, $\hat{e}_2$ are tangent locally to the ξ = const. and η = const. traces on ζ = const., respectively. The unit vector $\hat{e}_3$ lies in the direction of $\hat{e}_1 \times \hat{e}_2$ ($\perp$ to the plane ζ).

The intersection of ξ, η lies on a circle (at z = const.) with the unit tangent vector $\hat{e}_3$. The $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ vectors change direction according to P. The vectors $\nabla\xi_1$, $\nabla\xi_2$, $\nabla\xi_3$ are $\perp$ to ξ_1, ξ_2, ξ_3, respectively; therefore, $\hat{e}_3 = h_3 \nabla \xi_3$. If now $d\bar{r}_3$ is tangent to the ξ_3 circle, $|d\bar{r}_3| = dS_3$, then $d\bar{r}_3 \cdot \hat{e}_3 = dS_3 = d\bar{r}_3 \cdot h_3 \nabla\xi_3$, or $dS_3 = h_3 d\xi_3$. Similarly, $\hat{e}_2 = h_2 \nabla \xi_2$, $\hat{e}_1 = h_1 \nabla\xi_1$, or $dS_2 = h_2 d\xi_2$, $dS_1 = h_1 d\xi_1$. The unit vectors are such that $\hat{e}_1 = \hat{e}_2 \times \hat{e}_3$, $\hat{e}_2 = \hat{e}_3 \times \hat{e}_1$, and $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$.

In this coordinate system $x = -\cosh\xi \cos\eta \sin\zeta$, $y = C \cosh\xi \cos\eta \cos\zeta$, $z = C \sinh\xi \sin\eta$; in a plane ζ = const., the coordinate surfaces ξ, η are of the form

$$\frac{z^2}{C^2 \sinh^2\xi} + \frac{\rho^2}{C^2 \cosh^2\xi} = 1 \quad , \quad \frac{\rho^2}{C^2 \cos^2\eta} - \frac{z^2}{C^2 \sin^2\eta} = 1 \qquad (10)$$

where $\rho^2 = x^2 + y^2$, $\zeta = 0$ corresponds to $x = 0$, and $\zeta = \pi/2$ to $y = 0$. In order to transform the (ξ, η, ζ) system (Eqs. 5 through 9) into a coordinate system (σ, τ, π) in which the shock, body, stagnation streamline, and sonic surface are known in terms of values of the new coordinates in the manner of Ref. 1 (the method of Ref. 1 pertains to axisymmetric elliptic regions), stream functions (Refs. 2 and 3) ψ, ϕ are introduced. Note that the mass flow $\dot{m}_\infty$, through a portion Λ_n of a surface, Λ, covered by curvilinear coordinates, α and β, (with metrics, h and k) is

$$\dot{m}_\infty = \iint_{\Lambda_n} (\nabla\psi \times \nabla\phi) \cdot \overline{n} dS$$

$$= \iint_{\Lambda_n} [(h\psi_\alpha \overline{e}_1 + k\psi_\beta \overline{e}_2) \times (h\phi_\alpha \overline{e}_1 + k\phi_\beta \overline{e}_2)] \cdot \frac{(-\overline{e}_3)}{hk} \, d\alpha d\beta$$

$$= (\psi_2 - \psi_1)(\phi_2 - \phi_1) \tag{11}$$

therefore, for a plane $-x =$ const. $(\perp \overline{V}_\infty)$ divided into rectangular "boxes" by $z =$ const. and $y =$ const. planes, $\rho_\infty V_\infty (z_n - z_{n-1})(y_k - y_{k-1}) = (\psi_n - \psi_{n-1})(\phi_k - \phi_{k-1})$ represents the $\dot{m}$ in terms of the area of a particular "box." For plane flow $(y_k - y_{k-1}) = 1$, $(\phi_k - \phi_{k-1}) = 1$, and (Eq. 11) becomes $\dot{m}_\infty = (\psi_2 - \psi_1)$.

Consider next the independent variable transformation

$$\left(\alpha \frac{\tau}{\sin \eta}\right) = \frac{\psi}{\sinh \xi \sin \eta} \quad , \quad \left(\beta \frac{\sigma}{\cos \zeta}\right) = \frac{\phi}{\cosh \xi \cos \eta \cos \zeta} \quad ,$$

$$\pi = \alpha\beta \left(\frac{\sigma}{\sin \eta}\right)\left(\frac{\tau}{\cos \zeta}\right) \sinh \xi \tag{12}$$

or

$$\alpha' \frac{\tau}{\xi_2} = \frac{\psi}{\xi_1 \xi_2} \quad , \quad \left[\beta' \frac{\sigma}{\left(1 - \xi_3^2\right)^{1/2}}\right] = \frac{\phi}{\left[\left(C^2 + \xi_1^2\right)\left(1 - \xi_2^2\right)\left(1 - \xi_3^2\right)\right]^{1/2}} \quad ,$$

$$\pi = \alpha'\beta' \frac{\sigma}{\xi_2} \frac{\tau}{\xi_3} \xi_1 \tag{12a}$$

where α, β, α', β' are constants. Through Eq. (12), the region $R_1 \cap \Lambda$ transforms to $\tilde{R}_1 \cap \tilde{\Lambda}$; that this is true will be demonstrated after the following preliminary remarks leading to a method based on Eq. (12) and generalizations of the mass flow integral Eq. (11).

For certain of the more elementary methods of the first type, it is necessary that normal derivatives $h_1^{-1}\partial u/\partial\xi_1$, $h_1^{-1}\partial v/\partial\xi_1$, $h_1^{-1}\partial w/\partial\xi_1$, $h_1^{-1}\partial p/\partial\xi_1$, and $h_1^{-1}\partial\rho/\partial\xi_1$ (and higher derivatives) to the ξ_s surface be determined in terms of known tangential derivatives and evaluated from shock data. In this manner, provided a high accuracy difference scheme is coupled into the method, data is found on an adjacent ξ surface to the shock after multiplying by a suitable $\Delta\xi$ value. Repeating the evaluation of the derivatives, the entire flow field is then determined, provided certain finite difference conditions on optimum step size, stability, systematic errors, truncation error, etc., are satisfied.[4] These conditions have

[4] Straightforward methods of this type for three-dimensional flow will invariably experience numerical difficulties from growth of round-off error, etc., and require progressively more non-central high-order difference schemes to be used in conjunction with the ξ set of derivatives near the sonic surface, therefore, implying a much increased manipulation of numbers (and techniques) with no guarantee of convergence to sonic line values.

only been theoretically determined for the improperly posed Laplace equation (Refs. 13 and 14); however, it is assumed that the underlying principle of these more simple cases carries over to the elliptic non-linear system of equations. A more satisfactory approach is based on certain transformations (Refs. 1 and 2) initially made on the PDE system in order to reduce numerical difficulties associated with the above method. In this method known sonic line boundary conditions are naturally introduced, and, in addition, flow boundaries and streamlines are known in the transformed system permitting a reduction in the order of the transformed system. Application of known sonic line boundary conditions ($M_{\infty} >> 1$) are generally absent from methods of the first type. As both methods essentially involve marching methods, clearly generalized considerations of Refs. 13 and 14 are of significance. Other types of solutions for three-dimensional subsonic flows are suggested by Ref. 3, in which systems of nonlinear ordinary equations for terms of a series expansion for density and stream function pairs about the flow axis are derived; further considerations to solutions of this nature will not be given in this report, however.

For the first method, normal derivatives from Eqs. (5) through (9) are

$$\begin{pmatrix} \frac{\partial u}{h_1 \partial \xi_1} \\ \frac{\partial v}{h_1 \partial \xi_1} \\ \frac{\partial w}{h_1 \partial \xi_1} \\ \frac{\partial p}{h_1 \partial \xi_1} \\ \frac{\partial \rho}{h_1 \partial \xi_1} \end{pmatrix} = -N^{-1} \left[A \begin{pmatrix} \frac{\partial u}{h_2 \partial \xi_2} \\ \frac{\partial v}{h_2 \partial \xi_2} \\ \frac{\partial w}{h_2 \partial \xi_2} \\ \frac{\partial p}{h_2 \partial \xi_2} \\ \frac{\partial \rho}{h_2 \partial \xi_2} \end{pmatrix} + B \begin{pmatrix} \frac{\partial u}{h_3 \partial \xi_3} \\ \frac{\partial v}{h_3 \partial \xi_3} \\ \frac{\partial w}{h_3 \partial \xi_3} \\ \frac{\partial p}{h_3 \partial \xi_3} \\ \frac{\partial \rho}{h_3 \partial \xi_3} \end{pmatrix} + C \begin{pmatrix} u \\ v \\ w \\ p \\ \rho \end{pmatrix} \right] , \qquad (5a\text{-}8a)$$

$$|N| \neq 0 \text{ in } R_1 \cap \Lambda \text{ for } M < 1 \qquad (9a)$$

where

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & u/\rho \\ 0 & 0 & 0 & u\rho^{-\gamma} & -u\gamma p \rho^{-\gamma-1} \\ u & 0 & 0 & 1/\rho & 0 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & v/\rho \\ 0 & 0 & 0 & v\rho^{-\gamma} & -v\gamma p \rho^{-\gamma-1} \\ v & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 1/\rho & 0 \\ 0 & 0 & v & 0 & 0 \end{bmatrix}$$

$$
B = \begin{bmatrix} 0 & 0 & 1 & 0 & w/\rho \\ 0 & 0 & 0 & w\rho^{-\gamma} & w \\ w & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 1/\rho & 0 \end{bmatrix} .
$$

$$
C = \begin{bmatrix} \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_1}(h_2h_3) & \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_2}(h_1h_3) & \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_3}(h_1h_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h_1h_2}\left(u\frac{\partial h_1}{\partial\xi_2} - v\frac{\partial h_2}{\partial\xi_1}\right) & \frac{1}{h_1h_3}\left(u\frac{\partial h_1}{\partial\xi_3} - w\frac{\partial h_3}{\partial\xi_1}\right) & 0 & 0 \\ \frac{1}{h_1h_2}\left(v\frac{\partial h_2}{\partial\xi_1} - u\frac{\partial h_1}{\partial\xi_2}\right) & 0 & \frac{1}{h_2h_3}\left(v\frac{\partial h_2}{\partial\xi_3} - w\frac{\partial h_3}{\partial\xi_2}\right) & 0 & 0 \\ \frac{1}{h_1h_3}\left(w\frac{\partial h_3}{\partial\xi_1} - u\frac{\partial h_1}{\partial\xi_3}\right) & \frac{1}{h_2h_3}\left(w\frac{\partial h_3}{\partial\xi_2} - v\frac{\partial h_2}{\partial\xi_3}\right) & 0 & 0 & 0 \end{bmatrix}
$$

and boundary conditions on the prescribed ξ_s = const. shock are, for a perfect gas at constant γ, at a point $P(\xi_s, \eta, \zeta)$, for $\overline{V} = \sum_n \hat{e}_n h_n f^n$

$$
f^1 = \frac{u_s}{h_1} = \frac{\cos\eta \sin\zeta \sinh\xi_s}{h_1\left(\sinh^2\xi_s + \sin^2\eta\right)^{1/2}} \frac{(\gamma - 1)M^2_{\infty, N} + 2}{(\gamma + 1)M^2_{\infty, N}} \qquad (10a)
$$

$$
p_s = \left[1 + \frac{2\gamma}{\gamma + 1}\left(M^2_{\infty, N} - 1\right)\right] \qquad f^3 = \frac{w_s}{h_3} = -\frac{\cos\zeta}{h_3} \qquad (11a\text{-}12a)
$$

$$
f^2 = \frac{v_s}{h_2} = \frac{\sin\eta \sin\zeta \cosh\xi_s}{\left(\sinh^2\xi_s + \sin^2\eta\right)^{1/2}} \frac{1}{h_2} \qquad \rho_s = \frac{(\gamma + 1)M^2_{\infty, N}}{(\gamma - 1)M^2_{\infty, N} + 2}
$$

(13a-14a)

where $M_{\infty, N}^2 = M_\infty^2 \left[\cos^2 \eta \sin^2 \zeta \sinh^2 \xi_s / \left(\sinh^2 \xi_s + \sin^2 \eta\right)\right]$ and the corresponding entropy is

$$S_s - S_\infty = \log\left\{\left[1 + \frac{2\gamma}{\gamma + 1}\left(M_{\infty, N}^2 - 1\right)\right]^{\frac{1}{\gamma - 1}} \left[\frac{(\gamma + 1)M_{\infty, N}^2}{(\gamma - 1)M_{\infty, N}^2 + 2}\right]^{\frac{-\gamma}{\gamma + 1}}\right\} \tag{13}$$

while on the body, b, $S = S\big|_{max.} = \text{const.}$ (13a)

$$\left[\frac{d\xi_1\left(\cosh^2 \xi - \cos^2 \eta\right)^{1/2}}{u \cosh \xi}\right]\Bigg|_b = C\left[\frac{d\xi_2\left(\cosh^2 \xi - \cos^2 \eta\right)^{1/2}}{v \cos \eta}\right]\Bigg|_b$$

$$= \frac{d\xi_3 \cosh \xi \cos \eta}{w \cos \zeta} C\Bigg|_b \tag{14}$$

Suppose now that Eqs. (5) through (9) are subject to independent variable transformations (Eq. 12) by means of two-component stream functions, but not put into Eqs. (5a) through (9a) form, as the calculation scheme proposed does not require this form. Then conditions on the sonic surface may be determined by characteristics methods so that $R \cap \Lambda$ conditions are known on a closed boundary.

In that curvilinear coordinates are used throughout, differentiations with respect to coordinates are best done expressing vectors in their contravariant form (e.g., $\overline{V} = \sum_{n=1}^{3} \hat{e}_n h_n f^n$, where f^n are components of a contravariant vector); this provides a systematic formalism for the effects of differentiation of the "unit vectors." In general, then, for the velocity,

$$\frac{\partial \overline{V}}{\partial \xi_j} = \sum_{n=1}^{3} \hat{e}_n h_n \left[\frac{\partial f^n}{\partial \xi_j} + \sum_{m=1}^{3} f^m \left\{ {n \atop m\ j} \right\} \right] \tag{14b}$$

or the components of the contravariant vector which correspond to the derivative of the ordinary vector with respect to ξ_j are $f^i_{,j} = \frac{\partial f^i}{\partial \xi_j} + \sum_{m=1}^{3} f^m \left\{ {i \atop m\ j} \right\}$. The symbol $\left\{ {i \atop m\ j} \right\}$ denotes the Christoffel symbol of the second kind, generally formed from those of the first kind from $\left\{ {k \atop i\ j} \right\} = g^{k\alpha}[ij, \alpha]$, where $[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$, and $g^{k\alpha}$ is the contravariant tensor $g^{ij} = G^{ij}/|g_{ij}|$, G^{ij} is the cofactor of the element g_{ij} in $|g_{ij}|$. Provided transformations are next taken with the covariant law, then (for scalars), with Eq. (12a),

$$\begin{pmatrix} \frac{\partial(\ )}{\partial \xi_1} \\ \frac{\partial(\ )}{\partial \xi_2} \\ \frac{\partial(\ )}{\partial \xi_3} \end{pmatrix} = \begin{bmatrix} \frac{\partial \sigma}{\partial \xi_1} & \frac{\partial \tau}{\partial \xi_1} & \frac{\partial \pi}{\partial \xi_1} \\ \frac{\partial \sigma}{\partial \xi_2} & \frac{\partial \tau}{\partial \xi_2} & \frac{\partial \pi}{\partial \xi_2} \\ \frac{\partial \sigma}{\partial \xi_3} & \frac{\partial \tau}{\partial \xi_3} & \frac{\partial \pi}{\partial \xi_3} \end{bmatrix} \begin{pmatrix} \frac{\partial(\ )}{\partial \sigma} \\ \frac{\partial(\ )}{\partial \tau} \\ \frac{\partial(\ )}{\partial \pi} \end{pmatrix} \tag{15}$$

where the column vectors of J represent covariant vectors. In terms of these quantities, then, the derivatives of the boundary conditions (Eqs. 10a through 14a, and 13) may be expressed in terms of the Christoffel symbols, using Eqs. (4a), (4c), and (14b).

At the sonic surface,

$$f(\psi, \phi) = \rho^{1-\gamma} \frac{\frac{1}{M_\infty^2} + \frac{\gamma - 1}{2}}{\gamma\left(\frac{\gamma + 1}{2}\right)} \tag{16}$$

holds. Also, from Eq. (12a) at the shock

$$f(\psi, \phi) = \left\{ \left[1 - \frac{2\gamma}{\gamma + 1} M_\infty^2 \frac{\xi_{1s}^2\left(\xi_{1s}^2 - \frac{\psi^2}{K_1^2}\right)\left(1 - \frac{\xi_{1s}^2 \phi^2}{K_2^2\left(C^2 + \xi_{1s}^2\right)\left(\xi_{1s}^2 - \psi^2/K_1^2\right)}\right)}{\xi_{1s}^4 + C^2 \frac{\psi^2}{K_1^2}} \right] \right.$$

$$\left. \times \left[\frac{2 + (\gamma - 1)M_\infty^2 \frac{\xi_{1s}^2\left(\xi_{1s}^2 - \frac{\psi^2}{K_1^2}\right)\left(1 - \frac{\xi_{1s}^2 \phi^2}{K_2^2\left(C^2 + \xi_{1s}^2\right)\left(\xi_{1s}^2 - \psi^2/K_1^2\right)}\right)}{\left(\xi_{1s}^4 + C^2\psi^2/K_1^2\right)}}{(\gamma + 1)M_\infty^2 \frac{\xi_{1s}^2\left(\xi_{1s}^2 - \frac{\psi^2}{K_1^2}\right)\left(1 - \frac{\xi_{1s}^2 \phi^2}{K_2^2\left(C^2 + \xi_{1s}^2\right)\left(\xi_{1s}^2 - \psi^2/K_1^2\right)}\right)}{\left(\xi_{1s}^4 + \frac{C^2\psi^2}{K_1^2}\right)}} \right]^\gamma \right\} \tag{17}$$

The constants K_1, K_2 are defined at the shock by Eqs. (18) and (19); at the shock front in either coordinate system, (σ, τ, π) or (ξ_1, ξ_2, ξ_3), f is known from Eq. (12a) and the related inverse relations. The relation Eq. (17) holds throughout $(R_1 \cap \Lambda)$ and, when used numerically with Eq. (16), yields a method whereby the sonic surface may be computed (Refs. 1 and 3).

To describe the transformation (Eq. 12), consider the quantity $|\epsilon|$ as a small positive constant, $|\epsilon| << 1$. The quantities α, β, α', β' are also small constants, or scale factors, on τ, σ (Eq.. 12 and 12a). By construction of a suitable approximation surface (to be described later in this section) through the neighborhood of $z = 0$, $y = 0$ on the $-x$ = const. plane (stagnation point), and on the stagnation streamline though this point to the ξ_{1_s} = const. shock, ψ and ϕ are of the order of $|\epsilon|^2$ on this surface. On the body surface, where $\psi = 0$ and $\phi = 0$ simultaneously, the ψ, ϕ are of the order of $|\epsilon|$ on the approximation surface Ω in the neighborhood of any point on the body surface. Away from these regions, $\psi >> |\epsilon|$, $\phi >> |\epsilon|$ and may be considered simply as ψ, ϕ respectively. On the shock, from Eq.(12), assume $(\alpha\tau/\sin\eta)$ = const., $(\beta\sigma/\cos\zeta)$ = const. so that in terms of (σ, τ, π) coordinates, π = (const.) $\sinh\xi$. Then, as $\xi_1 = \xi_{1_s}$ = const., the prescribed three-dimensional shock in oblate-spheroidal coordinates transforms to π = (const.) in the new system (Fig. 2). As $0 < \eta < \pi/2$, $0 < \tau <$ const.; $0 < \zeta < \pi/2$, $0 < \sigma <$ const., outside Ω. On ξ_s, coordinates (ζ, η) determine a point; corresponding to this point is (τ, σ) on π = (const.). For a locally nonvanishing Jacobian, then, a portion of the shock $\Lambda(\xi_s, \eta, \zeta)$ may be mapped into a region $\Lambda'(\pi, \sigma, \tau)$ in the new coordinate system. Provided next the box-like regions (see Eq. 11 for simplified discussion) on the $-x$ = const. plane are projected onto the ξ_s = const. shock (Fig. 1), then on the shock surface

$$\psi \sim \sinh\xi \sin\eta \sim z \tag{18}$$

$$\phi \sim \cosh\xi \cos\eta \cos\zeta \sim y \tag{19}$$

or $\int_{\psi_{n-1}}^{\psi_n} d\psi \sim \int_{z_{n-1}}^{z_n} dz$ and $\int_{\phi_{k-1}}^{\phi_k} d\phi \sim \int_{y_{k-1}}^{y_k} dy$ for regions away from the Ω trace on ξ_s. The projected box-like regions do not, in general, conform to the curvilinear η, ζ system on the shock surface, ξ_s. Immediately downstream of the ξ_s shock, relations Eqs.(18) and (19) are no longer valid. For a streamline of the flow field (which lies along the intersection of ψ = const., ϕ = const. surfaces on a common value entropy (S/R) line imbedded in both surfaces), the conditions become ψ = const., ϕ = const. on a general streamline, or, from Eq.(12),

$$\alpha\tau = \frac{\text{const.}}{\sinh \xi} \quad , \quad \beta\sigma = \frac{\text{const.}}{\cosh \xi \cos \eta} \tag{20}$$

and in the neighborhood of the body

$$\alpha\tau = \frac{|\epsilon|/2}{\sinh \xi} \quad , \quad \beta\sigma = \frac{|\epsilon|/2}{\cosh \xi \cos \eta} \tag{21}$$

as $O(\psi) \sim O(|\epsilon|)$; $O(\phi) \sim O(|\epsilon|)$, near the part of Ω over the body.[5]

In that the stagnation streamline and body stream surface have an entropy $(S/R)_s$, it is necessary that a suitable representation be found for combinations of the new coordinates to represent the adjoining stagnation streamline and body as a surface with similar properties; to facilitate this representation, a surface consisting of a tubular region of radius $|\epsilon|^2$ surrounding the stagnation streamline joining a surface that envelopes

[5]Note that the regions of Ω are chosen such that the $|\epsilon|$ inequalities are satisfied for a definite, given $|\epsilon|$ value.

the body (within $|\epsilon|$ of the body) is assumed to represent the $(S/R)_s$ surface. In this manner a continuous approximate surface[5] is constructed that lies arbitrarily close to the actual stagnation streamline and body surface, by suitable choice of an $|\epsilon|$ value. In the limit as $|\epsilon| \to 0$, the actual stagnation streamline and body surface are exactly approached. Now in the region of the stagnation streamline, $O|\cos \zeta| \sim O(\eta) \sim O(|\epsilon|)$, $0 < \xi \leq \xi_s$ (as the focus of the coordinate system is assumed to be within the body) so that in the limit $|\epsilon| \to 0$, $\tau \to 0$, $\sigma \to 0$, and π appears of the order of magnitude $\alpha\beta \max\limits_{|\xi - \xi_s| < \delta} |\sinh \xi|$ in the region of the stagnation streamline just behind the shock (δ is a small positive constant). This, then, in (σ, τ, π) coordinates, appears to be the π axis, $\pi \leq (\text{const.}) \max\limits_{|\xi - \xi_s| < \delta} |\sinh \xi|$. In the vicinity of the body surface, $\xi > 0$, $\pi/2 > \eta > 0$, $\pi/2 < \zeta < 0$, away from the tubular region and near Ω. The transformation system near Ω is then,

$$\left(\frac{\alpha\tau}{\sin \eta}\right) \sinh \xi \sin \eta = |\epsilon| \tag{22}$$

$$\left(\frac{\beta\sigma}{\cos \zeta}\right) \cosh \xi \cos \eta \cos \zeta = |\epsilon| \tag{23}$$

$$\pi = \left(\frac{\alpha\tau}{\sin \eta}\right)\left(\frac{\beta\sigma}{\cos \zeta}\right) \sinh \xi \tag{24}$$

Suppose that $|\epsilon|^2$ is negligible with respect to $|\epsilon|$ and $|\epsilon|$ is taken to be a small positive constant in the neighborhood of the body. From Eqs. (22), (23), and (24), then, $O(\alpha\tau/\sin \eta) \sim O(|\epsilon|/m)$, $O(\sigma/\cos \zeta) \sim O(|\epsilon|/n)$, where n, m may be chosen >1 for large ξ, β and for conditions sufficiently far from the stagnation point region. Then $O(\pi) \sim O[(|\epsilon|^2/mn) \sinh \xi]$. On the approximate surface, then, for conditions applicable near the forward part of the stagnation streamline, part of the π axis is reproduced; further, for conditions on the body surface sufficiently far from the intersection point ($\eta > 0$, $\zeta > 0$, ξ large) the π surface becomes asymptotically close to the $\pi = 0$ plane. The region of Ω in the vicinity of the stagnation point, or the juncture of the two approximate surfaces is yet to be examined. Initially, consider the z, y planes tangent to the outside of the $|\epsilon|^2$ tube and those that describe a square within the tube on the plane -x = const.; the ψ, ϕ planes lying between these surfaces (in this approximation) then generate an infinity of (ψ, ϕ) pairs such that any ψ, ϕ pair in this region forms streamlines lying both on the $|\epsilon|^2$ tube and the approximation stream surface within $|\epsilon|$ of the body surface while still imbedded in Ω. On the approximation body surface, then, for a $|\epsilon| \neq 0$, specifying a point (ξ, η, ζ) leads to a point $(\sigma, \tau, \pi \approx 0)$ in the new coordinates for regions sufficiently far from the stagnation point. In the region near the stagnation point on Ω, suppose that $O(\xi) \sim O(|\epsilon|)$, then[6] $O(\pi) \sim O(\alpha\beta|\epsilon|)$. Here it is assumed that in this region the $O(\sin 0^-) \sim O(|\epsilon|)$, $O(\cos \pi/2^-) \sim O(|\epsilon|)$. The value of $\sinh \xi$ for $\xi >> 1$ is effectively of the order of $1/2\ e^{\xi}$; for ξ small, then, $O(\sinh \xi) \sim O(-1/2\ e^{-\xi})$, ($\xi > 0$). For $(\alpha\tau/\sin \eta)$ and $(\beta\sigma/\cos \zeta)$ of the

[6]Here it is assumed that the coordinate point $(0, \xi_2, \xi_3)$ lies in the region of the stagnation point; note also that the stagnation point lies interior to Ω in the neighborhood of that point.

order of $|\epsilon| \neq 0$ for $\eta > 0$, $\zeta < \pi/2$ on the body (Fig. 1), then $O(\pi) \sim O(|\epsilon|^2)$. The constants α, β may be suitably chosen according to the degree of magnification of the σ, τ, π coordinates desired for a particular problem.[7] For ψ, ϕ increasing, then, the value of π is >0, consistent with the streamlines passing between the shock and body in the (σ, τ, π) space. These streamlines originate at intersections of ψ = const., ϕ = const. surfaces on an $x < 0$ plane, or equivalently, the mappings of these intersection points onto the π = const. plane in the (σ, τ, π) space.

For ξ large, then, sinh ξ is effectively larger; therefore, the scale of the π coordinate is exaggerated. The surface corresponding to $\xi = 0$ is the surface of disc $r = C$; by requiring the stagnation point to be close to the focus of the coordinate system, then, the three parts of the approximation surfaces satisfy the properties previously designated. In this manner, the forward part of the stagnation streamline just past the shock transforms into part of the π-axis; the part of the stagnation streamline closer to the stagnation point lies on a surface $O(\pi) \sim O(|\epsilon|)$, and the surface covering the body within $|\epsilon|$ possesses the property that $O(\pi) \sim O(|\epsilon|)$. In the quadrant of the (σ, τ, π) space into which the quadrant of the physical (ξ, η, ζ) shock layer space has been mapped, this surface is then of the general nature of a hyperbola of revolution with the π, σ, and τ axes as asymptotes. Note that from Eqs. (16) and (17), and the

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[7] Obviously the magnitudes of these constants are determined from geometrical considerations based on the chosen ξ_s shock, M_∞, C, etc.

results of Section IV, the sonic surface may be constructed; the closed region $\widetilde{R}_1 \cap \widetilde{\Lambda}$ therefore consists of the given shock, known sonic surface, and Ω in (σ, τ, π) space (Fig. 2). From Section IV it is also possible to construct stream surfaces, a priori, through this region. Note, also, that for transformation of vector quantities, Eq. (14b) must necessarily modify the form of Eq. (15) due to the presence of contributions from the differentiated unit vectors. This, then, roughly describes the bounding surfaces of the physical shock layer in terms of surfaces of the (σ, τ, π) space, provided certain assumptions are made concerning approximation surfaces to the actual $(S/R)_s$ surfaces. The Jacobian of the transformation (for scalars) is

$$\begin{bmatrix} \frac{\partial \sigma}{\partial \xi_1} & \frac{\partial \sigma}{\partial \xi_2} & \frac{\partial \sigma}{\partial \xi_3} \\ \frac{\partial \tau}{\partial \xi_1} & \frac{\partial \tau}{\partial \xi_2} & \frac{\partial \tau}{\partial \xi_3} \\ \frac{\partial \pi}{\partial \xi_1} & \frac{\partial \pi}{\partial \xi_2} & \frac{\partial \pi}{\partial \xi_3} \end{bmatrix} = J \quad ; \quad \sigma, \tau, \pi \epsilon C^2 \tag{25}$$

and J and the inverse $j = J^{-1}$ are to be values bounded away from 0, ∞ in the region bounded by the shock, sonic surface, and the continuous surface enveloping the stagnation streamline and body, in order to guarantee a continuous one-to-one mapping of points in one space to the other. This condition may require adjustment of the arbitrary constant values introduced throughout this Section.

III. THREE-DIMENSIONAL STREAM FUNCTIONS OF CLEBSCH AND MORE GENERAL TYPES

For the inviscid three-dimensional rotational flow of a compressible gas in a oblate spheroidal coordinate system, the streamline equations are obtained as integral curves of the energy equation

$$u \frac{\cosh \xi}{\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \frac{\partial}{\partial \xi_1}\left(\frac{p}{\rho^\gamma}\right) + \frac{v}{C} \frac{\cos \eta}{\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \frac{\partial}{\partial \xi_2}\left(\frac{p}{\rho^\gamma}\right)$$

$$+ \frac{w}{C} \frac{\cos \zeta}{\cosh \xi \cos \eta} \frac{\partial}{\partial \xi_3}\left(\frac{p}{\rho^\gamma}\right) = 0 \quad (26)$$

as

$$\frac{d\xi_1\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}}{u \cosh \xi} = C \frac{d\xi_2\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}}{v \cos \eta}$$

$$= C \frac{d\xi_3 \cosh \xi \cos \eta}{w \cos \zeta} \quad (27)$$

The general integral of Eq. (26) is of the form $\rho/\rho^\gamma = f[\psi(\xi, \eta, \zeta, u, v, w), \phi(\xi, \eta, \zeta, u, v, w)]$ where f is determined[8] from initial values of ρ/p^γ on the streamline intersection of ψ = const., ϕ = const. surfaces with a shock,

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[8] See Eq. (17).

here taken as a portion of the coordinate surface ξ = const. for $x < 0$. In that both ψ and ϕ satisfy Eq. (26) individually, it follows that

$$u \frac{\cosh \xi}{\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \psi_{\xi_1} + v \frac{\cos \eta}{C\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \psi_{\xi_2} + \frac{w}{C} \frac{\cos \zeta}{\cosh \xi \cos \eta} \psi_{\xi_3} = 0 \tag{28}$$

and

$$u \frac{\cosh \xi}{\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \phi_{\xi_1} + v \frac{\cos \eta}{C\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \phi_{\xi_2} + \frac{w}{C} \frac{\cos \zeta}{\cosh \xi \cos \eta} \phi_{\xi_3} = 0 \tag{29}$$

Solutions u, v, w of Eqs. (28) and (29) are determined to within a proportionality factor: by requiring these values to satisfy the **equation of continuity**

$$\frac{\partial}{\partial \xi_1}\left[\rho\mu\left(\frac{C^2 \cosh \xi}{\cos \zeta}\right)^{-1}\left(\cosh^2 \xi - \cos^2 \eta\right)^{-1/2}\right] + \frac{\partial}{\partial \xi_2}\left[\rho v\left(\frac{\cos \eta}{\cos \zeta}\right)^{-1}\left(\cosh^2 \xi - \cos^2 \eta\right)^{-1/2}\right] + \frac{\partial}{\partial \xi_3}\left[\rho w\left(C\frac{\cosh^2 \xi - \cos^2 \eta}{\cosh \xi \cos \eta}\right)^{-1}\right] = 0 \tag{30}$$

it follows that u, v, w are determined[9] in terms of the ψ, ϕ stream function pair as

$$u = \frac{1}{\rho}\left[\frac{\cos\zeta}{C^2 \cosh\xi}\left(\frac{1}{\cosh^2\xi - \cos^2\eta}\right)^{1/2}\right]^{-1} \left(\psi_{\xi_2}\phi_{\xi_3} - \psi_{\xi_3}\phi_{\xi_2}\right) \quad (31a)$$

$$v = \frac{1}{\rho}\left[-\frac{\cos\zeta}{\cos\eta}\left(\frac{1}{\cosh^2\xi - \cos^2\eta}\right)^{1/2}\right]^{-1} \left(\psi_{\xi_1}\phi_{\xi_3} - \psi_{\xi_3}\phi_{\xi_1}\right) \quad (31b)$$

$$w = \left[\frac{1}{\rho}\,\frac{\cos\eta\cosh\xi}{C(\cosh^2\xi - \cos^2\eta)}\right]^{-1} \left(\psi_{\xi_1}\phi_{\xi_2} - \psi_{\xi_2}\phi_{\xi_1}\right) \quad (31c)$$

which are the components of $\overline{V} = 1/\rho(\nabla\psi x \nabla\phi)$. In order to transform the basic quasi-linear first-order system (Section I) into another first-order system by means of dependent variable transformations based on Eq. (31), it is necessary that inverse relations of the form

$$\psi_{\xi_1} = \psi_{\xi_1}(\xi_1, \xi_2, \xi_3, u, v, w, \rho) \quad , \quad \psi_{\xi_2} = \psi_{\xi_2}(\xi_1, \xi_2, \xi_3, u, v, w, \rho) \quad ,$$

$$\psi_{\xi_3} = \psi_{\xi_3}(\xi_1, \xi_2, \xi_3, u, v, w, \rho) \quad , \quad \phi_{\xi_1} = \phi_{\xi_1}(\xi_1, \xi_2, \xi_3, u, v, w, \rho) \quad ,$$

$$\phi_{\xi_2} = \phi_{\xi_2}(\xi_1, \xi_2, \xi_3, u, v, w, \rho) \quad , \quad \phi_{\xi_3} = \phi_{\xi_3}(\xi_1, \xi_2, \xi_3, u, v, w, \rho) \quad ,$$

[9] Alternatively, $\nabla\phi = \nabla\psi x(-\rho/|\nabla\psi|^2)\overline{V}$; note that if ψ is given, then

$$\phi = -\int \frac{(\nabla\psi x \rho \overline{V}) \cdot d\overline{x}}{\nabla\psi \cdot \nabla\psi}$$

are known. Clearly the system (Eq. 31) is not sufficient for this purpose as implicit function theory yields only relations (for example) of the form $\psi_{\xi_2} \cos \eta \partial\psi_{\xi_3}/\partial(\rho u) - C\psi_{\xi_1} \cosh \xi \partial\psi_{\xi_3}/\partial(\rho v) = 0$, which lead to general integrals only reasserting conditions Eqs. (28) and (29). Additional assumptions are, therefore, necessary for determination of suitable inverse relations from Eqs. (31a), (31b), and (31c), provided this system is to be used to reduce Eqs. (5) through (9).

Transformations of the type $\overline{V} = 1/\rho(\nabla\psi x \nabla\phi)$ are Clebsch transformations. An alternative type is

$$\rho u h_2 h_3 = \frac{\partial\psi}{\partial\xi_2} \tag{32}$$

$$\rho v h_1 h_3 = -\left(\frac{\partial\psi}{\partial\xi_1} + \frac{\partial\phi}{\partial\xi_3}\right) \tag{33}$$

$$\rho w h_1 h_2 = \frac{\partial\phi}{\partial\xi_2} \tag{34}$$

where the metrics are as given in Eq. (4c). Provided $\rho\overline{V} = \text{curl}\ \overline{A}$ then the existence of real ψ, ϕ surfaces in $(R \cap \Lambda)$ becomes the problem of determining real values[10] for

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[10] For a $\nabla x \overline{B} = \nabla\psi x \nabla\phi$, $B_1 2h_1 = \psi\phi_{\xi_1} - \phi\psi_{\xi_1}$, $B_2 2h_2 = \psi\phi_{\xi_2} - \phi\psi_{\xi_2}$, $B_3 2h_3 = \psi\phi_{\xi_3}' - \phi\psi_{\xi_3}$, illustrating the difficulty associated with a form corresponding to Eqs. (35) and (36) for this transformation. The forms Eqs. (35) and (36) are nonunique, however, as A_i terms are determinable to within an arbitrary gradient.

$$\psi = h_3 A_3 - \int \frac{\partial}{\partial \xi_3} (h_2 A_2)\, d\xi_2 \tag{35}$$

$$\xi_1, \xi_2, \xi_3 \epsilon (R_1 \cap A)$$

$$\phi = - h_1 A_1 + \int \frac{\partial}{\partial \xi_1} (h_2 A_2)\, d\xi_2 \tag{36}$$

For the ψ, ϕ stream functions of Eqs. (32) through (34) to satisfy Eq. (26) individually, as (Eqs. 28 and 29), then

$$\frac{-h_1 \psi_{\xi_1}}{h_1 \phi_{\xi_1}} = \frac{h_2 \psi_{\xi_2}}{h_2 \phi_{\xi_2}} = \frac{h_3 \psi_{\xi_3}}{h_3 \phi_{\xi_3}} = \frac{uh_3}{wh_1} \tag{37a, b, c}$$

from Eqs. (32) through (34). This follows from the requirement that a point on a streamline is imbedded on both ϕ and ψ surfaces.

On the ψ = const. sheet and the ϕ = const. sheet, respectively, additional relations follow from

$$h_1^{-1} \psi_{\xi_1} d\xi_1 h_1 + h_2^{-1} \psi_{\xi_2} d\xi_2 h_2 + h_3^{-1} \psi_{\xi_3} d\xi_3 h_3 = 0 \tag{38}$$

$$h_1^{-1} \phi_{\xi_1} d\xi_1 h_1 + h_2^{-1} \phi_{\xi_2} d\xi_2 h_2 + h_3^{-1} \phi_{\xi_3} d\xi_3 h_3 = 0 \tag{39}$$

So that with Eqs. (28) and (29),

$$\frac{h_1 h_3^{-1} \psi_{\xi_1}}{\psi_{\xi_3}} = \frac{-h_3 d\xi_3 v h_1 h_2^2 + h_2 d\xi_2 w h_1 h_3^2}{h_1 d\xi_1 v h_2^2 h_3 - h_2 d\xi_2 u h_1^2 h_3} h_1 h_3^{-1} \tag{40}$$

$$\frac{h_2 h_3^{-1} \psi_{\xi_2}}{\psi_{\xi_3}} = \frac{-h_1 d\xi_1 w h_2 h_3^2 + h_3 d\xi_3 u h_1^2 h_2}{h_1 d\xi_1 v h_2^2 h_3 - h_2 d\xi_2 h_1^2 h_3} h_2 h_3^{-1} \tag{41}$$

$$\frac{h_1 h_3^{-1} \phi_{\xi_1}}{\phi_{\xi_3}} = \frac{-h_3 d\xi_3 v h_1 h_2^2 + h_2 d\xi_2 w h_1 h_3^2}{h_1 d\xi_1 v h_2^2 h_3 - h_2 d\xi_2 u h_1^2 h_3} h_1 h_3^{-1} \tag{42}$$

$$\frac{h_2 h_3^{-1} \phi_{\xi_2}}{\phi_{\xi_3}} = \frac{-h_1 d\xi_1 w h_2 h_3^2 + h_3 d\xi_3 h_1^2 h_2 u}{h_1 d\xi_1 v h_2^2 h_3 - h_2 d\xi_2 h_1^2 h_3 u} h_2 h_3^{-1} \tag{43}$$

on these constant valued surfaces. Also, for orthogonality of ψ, ϕ surfaces,

$$\frac{\phi_{\xi_1} \psi_{\xi_1}}{h_1^2} + \frac{\phi_{\xi_2} \psi_{\xi_2}}{h_2^2} + \frac{\phi_{\xi_3} \psi_{\xi_3}}{h_3^2} = 0 \tag{44}$$

which generally is satisfied only in the free stream and on the shock, but is probably too restrictive in R_1 and, therefore, will not be assumed (Fig. 1). Then it follows that in R_1, outside of Ω,

$$
\begin{bmatrix} \psi_{\xi_1} \\ \psi_{\xi_2} \\ \psi_{\xi_3} \\ \phi_{\xi_1} \\ \phi_{\xi_2} \\ \phi_{\xi_3} \end{bmatrix}
=
\begin{bmatrix}
\dfrac{(uh_3)(\rho wh_1h_2)_{\xi_1} + (\rho uh_2h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho wh_1h_2)_{\xi_2} - (\rho uh_2h_3)_{\xi_2}(wh_1)}\,\rho uh_2h_3(-1) \\[2ex]
\rho uh_2h_3 \\[2ex]
\dfrac{(uh_3)(\rho wh_1h_2)_{\xi_3} - (\rho uh_2h_3)_{\xi_3}(wh_1)}{(uh_3)(\rho wh_1h_2)_{\xi_2} - (wh_1)(\rho uh_2h_3)_{\xi_2}}\,\rho uh_2h_3 \\[2ex]
\dfrac{(uh_3)(\rho wh_1h_2)_{\xi_1} + (\rho uh_2h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho wh_1h_2)_{\xi_2} - (\rho uh_2h_3)_{\xi_2}(wh_1)}\,\rho wh_1h_2 \\[2ex]
\rho wh_1h_2 \\[2ex]
-\rho vh_1h_3 + \rho uh_2h_3\,\dfrac{(uh_3)(\rho wh_1h_2)_{\xi_1} + (\rho uh_2h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho wh_1h_2)_{\xi_2} - (\rho uh_2h_3)_{\xi_2}(wh_1)}
\end{bmatrix}
\tag{45}
$$

At the shock, these relations are explicitly given for the ψ, ϕ derivative column vectors.

Note that in Ref. 1, the expression Eq. (37) for the axisymmetric case is unnecessary, as $\psi_r\psi_x\delta/R - \psi_r\psi_x\delta/R = 0$; for the three-dimensional case, however, Eq. (37) is an additional relation that must be satisfied.

Note that in Eq. (45) the evaluation of derivatives is to be accomplished by expressing the velocities in their contravariant form, then operating with Eq. (14a), in part, for the complete derivative. This process has been incorporated into Eq. (45).

Note on the inverse relations:

The expressions in Eq. (45) contain certain derivative terms. Now consider n sets of values given a priori $u^{(1)}, v^{(1)}, w^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}$; $u^{(2)}, v^{(2)}, w^{(2)}, \xi_1^{(2)}, \xi_2^{(2)}, \xi_3^{(2)}$, etc., in Eq. (45) with metrics known and derivatives approximated by some discrete approximate expression involving (1), . . . , (n) quantities and the appropriate metrics. Then values $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)}$ may be calculated for the first set of values, respectively, and also $\sigma^{(1)}, \ldots, \sigma^{(n)}, \tau^{(1)}, \ldots, \tau^{(n)}$. In terms of these quantities, the partials may be represented accurately, locally, in the vicinity of a particular set of values by this "finite difference" type approach. In a certain small region, then, derivatives have a certain "best approximation," as a number of points have been used for their determination.

Now for two particular sets of values, (j), (j+1), with the best derivative approximation, a box bounded by planes $\pi^{(j)}$ = const., $\pi^{(j+1)}$ = const., $\sigma^{(j)}$ = const., $\sigma^{(j+1)}$ = const., $\tau^{(j)}$, $\tau^{(j+1)}$ in $\tilde{R}$ exists. In the open region N (for which these planes are the closure), then, the $u^{(k)}, v^{(k)}, w^{(k)} \in N$ are to be considered as parameters for the inversion of Eqs. (62) through (65) for a particular set of $u^{(t)}, v^{(t)}, w^{(t)}$ values $1 \leq t \leq n$. If next $u^{(t)}, v^{(t)}, w^{(t)}$ are varied and the process repeated, then the dependence upon the parameters can be ascertained by the variances in surface shape, average values within the box, etc. With this information the inverse relations suggested later can be performed numerically with respect to the u, v, w, ρ, parameters. See also Eqs. (62 through 65).

Consider the special case of plane flow in rectangular coordinates at constant density ($w = 0$, $w_z = 0$, $\rho = 1$, h_1, h_2, $h_3 = 1$, $\phi_2 = 0$), then

$$\begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \\ \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \begin{bmatrix} -v \\ u \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

Note also that for $\rho = 1$, $h_i = 1$, the irrotationality conditions for the stream function pair are of the form $\nabla^2\phi = 0$; $\nabla^2\psi = 0$ with $\phi_x = \psi_z$.

With these relations, transformation from the curvilinear (ξ_1, ξ_2, ξ_3) space to σ, τ, π coordinates by means of Eq. (15) is now possible. Now the metrics of the two systems are related by

$$ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2 = \frac{d\tau^2}{\left(\frac{\partial\tau}{\partial\xi_1}\right)^2 + \left(\frac{\partial\tau}{\partial\xi_2}\right)^2 + \left(\frac{\partial\tau}{\partial\xi_3}\right)^2}$$

$$+ \frac{d\sigma^2}{\left(\frac{\partial\sigma}{\partial\xi_1}\right)^2 + \left(\frac{\partial\sigma}{\partial\xi_2}\right)^2 + \left(\frac{\partial\sigma}{\partial\xi_3}\right)^2} + \frac{d\pi^2}{\left(\frac{\partial\pi}{\partial\xi_1}\right)^2 + \left(\frac{\partial\pi}{\partial\xi_2}\right)^2 + \left(\frac{\partial\pi}{\partial\xi_3}\right)^2}$$

$$= \tilde{h}_1^2 d\tau^2 + \tilde{h}_2^2 d\sigma^2 + \tilde{h}_3^2 d\pi^2 \tag{48}$$

and for orthogonality of the two coordinate systems,

$$\frac{\partial \xi_1}{\partial \sigma}\frac{\partial \xi_1}{\partial \tau} + \frac{\partial \xi_2}{\partial \sigma}\frac{\partial \xi_2}{\partial \tau} + \frac{\partial \xi_3}{\partial \sigma}\frac{\partial \xi_3}{\partial \tau} = 0 \tag{49}$$

$$\frac{\partial \xi_1}{\partial \sigma}\frac{\partial \xi_1}{\partial \pi} + \frac{\partial \xi_2}{\partial \sigma}\frac{\partial \xi_2}{\partial \pi} + \frac{\partial \xi_3}{\partial \sigma}\frac{\partial \xi_3}{\partial \pi} = 0 \tag{50}$$

$$\frac{\partial \xi_1}{\partial \tau}\frac{\partial \xi_1}{\partial \pi} + \frac{\partial \xi_2}{\partial \tau}\frac{\partial \xi_2}{\partial \pi} + \frac{\partial \xi_3}{\partial \tau}\frac{\partial \xi_3}{\partial \pi} = 0 \tag{51}$$

and

$$\frac{\partial \sigma}{\partial \xi_2}\frac{\partial \sigma}{\partial \xi_1} + \frac{\partial \tau}{\partial \xi_2}\frac{\partial \tau}{\partial \xi_1} + \frac{\partial \pi}{\partial \xi_2}\frac{\partial \pi}{\partial \xi_1} = 0 \tag{49a}$$

$$\frac{\partial \sigma}{\partial \xi_2}\frac{\partial \sigma}{\partial \xi_3} + \frac{\partial \tau}{\partial \xi_2}\frac{\partial \tau}{\partial \xi_3} + \frac{\partial \pi}{\partial \xi_2}\frac{\partial \pi}{\partial \xi_3} = 0 \tag{50a}$$

$$\frac{\partial \sigma}{\partial \xi_1}\frac{\partial \sigma}{\partial \xi_3} + \frac{\partial \tau}{\partial \xi_1}\frac{\partial \tau}{\partial \xi_3} + \frac{\partial \pi}{\partial \xi_1}\frac{\partial \pi}{\partial \xi_3} = 0 \tag{51a}$$

The elements of the J matrix (Eqs. 15 and 25) then become, with Eqs. (45) and (12A),

$$\frac{\partial \sigma}{\partial \xi_1} = -\frac{\sigma \xi_1}{\left(C^2 + \xi_1^2\right)} + \frac{\beta'^{-1} \rho w h_1 h_2}{\left[\left(1 - \xi_2^2\right)\left(C^2 + \xi_1^2\right)\right]^{1/2}}$$

$$\times \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \tag{52}$$

$$\frac{\partial \sigma}{\partial \xi_2} = \frac{\sigma \xi_2}{\left(1 - \xi_2^2\right)} + \frac{\beta'^{-1} \rho w h_1 h_2}{\left[\left(C^2 + \xi_1^2\right)\left(1 - \xi_2^2\right)\right]^{1/2}} \tag{53}$$

$$\frac{\partial \sigma}{\partial \xi_3} = \frac{-\beta'^{-1}}{\left[\left(C^2 + \xi_1^2\right)\left(1 - \xi_2^2\right)\right]^{1/2}}$$

$$\times \left[\rho v h_1 h_3 - \rho u h_2 h_3 \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \right] \tag{54}$$

$$\frac{\partial \tau}{\partial \xi_1} = \frac{1}{\xi_1} \left[- \tau - a'^{-1} \rho u h_2 h_3 \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \right] \tag{55}$$

$$\frac{\partial \tau}{\partial \xi_2} = \frac{a'^{-1}}{\xi_1} \rho u h_2 h_3 \tag{56}$$

$$\frac{\partial \tau}{\partial \xi_3} = \frac{a'^{-1}}{\xi_1} \rho u h_2 h_3 \frac{(uh_3)(\rho w h_1 h_2)_{\xi_3} - (\rho u h_2 h_3)_{\xi_3}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (wh_1)(\rho u h_2 h_3)_{\xi_2}} \tag{57}$$

$$\frac{\partial \pi}{\partial \xi_1} = \frac{\pi}{\xi_1} + \frac{\pi}{\tau} \frac{\partial \tau}{\partial \xi_1} + \frac{\pi}{\sigma} \frac{\partial \sigma}{\partial \xi_1} \tag{58}$$

$$\frac{\partial \pi}{\partial \xi_2} = - \frac{\pi}{\xi_2} + \frac{\pi}{\tau} \frac{\partial \tau}{\partial \xi_2} + \frac{\pi}{\sigma} \frac{\partial \sigma}{\partial \xi_2} \tag{59}$$

$$\frac{\partial \pi}{\partial \xi_3} = - \frac{\pi}{\xi_3} + \frac{\pi}{\tau} \frac{\partial \tau}{\partial \xi_3} + \frac{\pi}{\sigma} \frac{\partial \sigma}{\partial \xi_3} \tag{60}$$

in which coupling exists between the (ξ_1, ξ_2, ξ_3) and (σ, τ, π) systems. Note that the order of the system has been preserved as Eq. (45) is known, and the Eqs. (52) through (60) are ψ, ϕ independent. Now the space R (locally Euclidean) formed by ξ_1, ξ_2, ξ_3 is to map (locally) onto the totally Euclidean space $\tilde{R}$ (σ, τ, π) in Fig. 2. For the totally Euclidean[11] space of (σ, τ, π), it is assumed

$$\tilde{h}_1^2 = K_1^{-1} = \text{const.} > 0 \quad , \qquad \tilde{h}_2^2 = K_2^{-1} = \text{const.} > 0 \quad ,$$

$$\tilde{h}_3^2 = K_3^{-1} = \text{const.} > 0 \tag{61}$$

so that all components of the curvature tensor are zero throughout $\tilde{R}$. Then, from Eqs. (48) and (52) through (60),

$$\tau = \alpha'^{-1}(\rho u h_2 h_3) \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} (-1)$$

$$\pm \left[K_1 \xi_1^2 - \alpha'^{-2}(\rho u h_2 h_3)^2 \right.$$

$$\left. \times \left\{ 1 + \left[\frac{(uh_3)(\rho w h_1 h_2)_{\xi_3} - (\rho u h_2 h_3)_{\xi_3}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (wh_1)(\rho u h_2 h_3)_{\xi_2}} \right]^2 \right\} \right]^{1/2} \tag{62}$$

[11] All the Christoffel symbols vanish for constant metrics.

here K_1 is chosen such that

$$K_1 a'^2 > \xi_1^{-1} \rho u h_2 h_3 \left\{ 1 + \left[\frac{(uh_3)(\rho w h_1 h_2)_{\xi_3} - (\rho u h_2 h_3)_{\xi_3}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (wh_1)(\rho u h_2 h_3)_{\xi_2}} \right]^2 \right\} ,$$

throughout R, and (arbitrarily), $\tau \geq 0$ in the mapped octant. The constants here also are to be such that Eq. (25) conditions are met.

Similarly,

$$\sigma = - \frac{\beta'^{-1} \rho u h_2 h_3}{\left[(1 - \xi_2^2)(C^2 + \xi_1^2)\right]^{1/2}} \left[\frac{\xi_2}{1 - \xi_2^2} + \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \cdot \frac{\xi_1}{C^2 + \xi_1^2} \right]$$

$$\times \left[\frac{\xi_1^2}{(C^2 + \xi_1^2)^2} + \frac{\xi_2^2}{(1 - \xi_2^2)^2} \right]^{-1} \pm \left[\frac{\xi_1^2}{(C^2 + \xi_1^2)^2} + \frac{\xi_2^2}{(1 - \xi_2^2)^2} \right]^{-1}$$

$$\times \left\{ \frac{\beta'^{-2}(\rho u h_2 h_3)^2}{(C^2 + \xi_1^2)(1 - \xi_2^2)} \left[\frac{\xi_2}{1 - \xi_2^2} + \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \frac{\xi_1}{C^2 + \xi_1^2} \right]^2 - \Omega' \right\}^{1/2} \quad \textbf{(63)}$$

where

$$\Omega' = \left[\frac{\xi_1^2}{(C^2 + \xi_1^2)^2} + \frac{\xi_2^2}{(1 - \xi_2^2)^2} \right] \left\{ \frac{\beta'^{-2}(\rho u h_2 h_3)^2}{(1 - \xi_2^2)(C^2 + \xi_1^2)} \left[\frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \right] \right\}^2$$

$$+ \frac{\beta'^{-2}(\rho u h_2 h_3)^2}{(C^2 + \xi_1^2)(1 - \xi_2^2)} - K_2$$

$$+ \frac{\beta'^{-2}}{(C^2 + \xi_1^2)(1 - \xi_2^2)} \left\{ \rho v h_1 h_3 - \rho u h_2 h_3 \left[\frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \right] \right\}^2 \quad \textbf{(64)}$$

Again, K_2 is chosen such that σ is real in $\widetilde{R}$, and $\sigma \geq 0$ in the transformed octant, representing the physical octant in R. In the same manner as the above, with (51a)

$$\pi = \pm\left\{\left[\left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2} + \frac{1}{\xi_3^2}\right) + \frac{2}{\tau}\left(\frac{\partial\tau}{\xi_1\partial\xi_1} - \frac{\partial\tau}{\xi_2\partial\xi_2} - \frac{\partial\tau}{\xi_3\partial\xi_3} + \frac{K_1}{2\tau}\right)\right.\right.$$

$$\left.\left. + \frac{2}{\sigma}\left(\frac{\partial\sigma}{\xi_1\partial\xi_1} + \frac{\partial\sigma}{\xi_2\partial\xi_2} + \frac{\partial\sigma}{\xi_3\partial\xi_3} + \frac{K_2}{2\sigma}\right)\right]K_3^{-1}\right\}^{-1/2} \qquad (65)$$

where the above partials are to be evaluated from Eqs. (62) through (64), and (52) through (57) expressions. For values of u, v, w, then $\tau = \tau(\xi_1, \xi_2, \xi_3;\ u, v, w)$, $\sigma = \sigma(\xi_1, \xi_2, \xi_3;\ u, v, w)$ and $\pi = \pi(\xi_1, \xi_2, \xi_3; u, v, w)$; provided, for fixed u, v, w at a point, $J \neq 0$, $j \neq 0$ and $\sigma, \tau, \pi \epsilon C^2$, and σ, τ, π are well defined for all $\xi_1, \xi_2, \xi_3 \epsilon R_1$, then an inverse exists for fixed u, v, w, i. e., $\xi_1 = \xi_1(\sigma, \tau, \pi; u, v, w)$, $\xi_2 = \xi_2(\sigma, \tau, \pi; u, v, w)$, and $\xi_3 = \xi_3(\sigma, \tau, \pi; u, v, w)$. This procedure was suggested earlier in Section III; note also that ρ may be included in the same category as u, v, w, in this inversion.

The constants K_1, K_2, K_3 change scale sufficiently in the (σ, τ, π) system such that real values of these coordinates are defined for real values of ξ_1, ξ_2, ξ_3 coordinates. The complexity of the suggested inverse relations clearly suggests numerical treatment. Here, $\pi > 0$, and K_3 is chosen so that π is always real. From Eq. (12A), in terms of (ξ_1, ξ_2, ξ_3) coordinates,

$$\psi = \xi_1(\rho u h_2 h_3) \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)}(-1)$$

$$\pm a'\xi_1 \left[K_1 \xi_1^2 - a'^{-2}(\rho u h_2 h_3)^2 \left\{ 1 + \left[\frac{(uh_3)(\rho w h_1 h_2)_{\xi_3} - (\rho u h_2 h_3)_{\xi_3}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (wh_1)(\rho u h_2 h_3)_{\xi_2}} \right]^2 \right\} \right]^{1/2} \tag{66}$$

and

$$\phi = -\left[\frac{\xi_1^2}{\left(C^2 + \xi_1^2\right)^2} + \frac{\xi_2^2}{\left(1 - \xi_2^2\right)^2} \right]^{-1} \times$$

$$\left[\rho u h_2 h_3 \left\{ \frac{\xi_2}{1 - \xi_2^2} \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \frac{-\xi_1}{C^2 + \xi_1^2} \right\} \right.$$

$$\pm \left\{ (\rho u h_2 h_3)^2 \left[\frac{\xi_2}{1 - \xi_2^2} - \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \frac{\xi_1}{C^2 + \xi_1^2} \right]^2 \right.$$

$$\left. \left. - \beta'^{-2}\left(C^2 + \xi_1^2\right)\left(1 - \xi_2^2\right)\Omega' \right\}^{1/2} \right] \tag{67}$$

which is of the indicated form of Eqs. (35) and (36) (the A_1, A_2, A_3 functions have $\xi_1, \xi_2, \xi_3, u, v, w$ as arguments), therefore, providing a means of construction of the ψ, ϕ surfaces.

IV. THE TRANSFORMED SPACE AND THE RESULTING SYSTEM OF EQUATIONS

In the prior section, the inverse relations necessary for Eqs. (5) through (9) to be written entirely in terms of (σ, τ, π) coordinates involved an approximate inversion of Eqs. (62) through (65) with u, v, w, ρ, held fixed. The transformed system is, therefore, determined to within as close an approximation as desired in a neighborhood of a point in (σ, τ, π) coordinates mapped from (ξ_1, ξ_2, ξ_3) coordinates by Eq. (12) or (12a). For a numerical marching procedure involving finite difference methods, on the basis of the transformed equations, the additional calculational error due to systematic errors for the approximately determined functions in small regions may be made to possess a small effect in total error by suitable adjustment of mesh size, etc., in the finite difference approach.

At a point in $\tilde{R}_1$ at which velocities $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ exist, then it follows that, with $h_1 = h_1(\sigma, \tau, \pi; u, v, w, \rho)$, $h_2 = h_2(\sigma, \tau, \pi; u, v, w, \rho)$, $h_3 = h_3(\sigma, \tau, \pi; u, v, w, \rho)$,

$$\tilde{u} = \frac{\tilde{h}_1 u}{h_1 \xi_1}\left\{ -\tau - a'^{-1} \rho u h_2 h_3 \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)} \right\}$$

$$+ \frac{a'^{-1}\tilde{h}_1 v \rho u h_2 h_3}{h_2 \xi_1} - \frac{\tilde{h}_1 w a'^{-1} \rho u h_2 h_3}{h_3 \xi_1} \frac{(uh_3)(\rho w h_1 h_2)_{\xi_3} - (\rho u h_2 h_3)_{\xi_3}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (wh_1)(\rho u h_2 h_3)_{\xi_2}} \qquad (68)$$

$$\tilde{v} = \frac{\tilde{h}_2 u}{h_1}\left\{\frac{-\sigma\xi_1}{C^2+\xi_1^2} + \frac{-\beta'^{-1}\rho u h_2 h_3}{\left[\left(1-\xi_2^2\right)\left(C^2+\xi_1^2\right)\right]^{1/2}} \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)}\right\}$$

$$+ \frac{\tilde{h}_2 v}{h_2}\left\{\frac{\sigma\xi_2}{\left(1-\xi_2^2\right)} + \frac{\beta'^{-1}\rho u h_2 h_3}{\left[\left(C^2+\xi_1^2\right)\left(1-\xi_2^2\right)\right]^{1/2}}\right\} - \frac{\tilde{h}_2 w \beta'^{-1}}{h_3\left[\left(C^2+\xi_1^2\right)\left(1-\xi_2^2\right)\right]^{1/2}}$$

$$\times\left\{\rho v h_1 h_3 - \rho u h_2 h_3 \frac{(uh_3)(\rho w h_1 h_2)_{\xi_1} + (\rho u h_2 h_3)_{\xi_1}(wh_1)}{(uh_3)(\rho w h_1 h_2)_{\xi_2} - (\rho u h_2 h_3)_{\xi_2}(wh_1)}\right\} \tag{69}$$

$$\tilde{w} = \frac{\tilde{h}_3 u\pi}{h_1}\left(\frac{1}{\xi_1} + \frac{1}{\tau}\frac{\partial\tau}{\partial\xi_1} + \frac{1}{\sigma}\frac{\partial\sigma}{\partial\xi_1}\right) + \frac{\tilde{h}_3 v\pi}{h_2}\left(-\frac{1}{\xi_2} + \frac{1}{\tau}\frac{\partial\tau}{\partial\xi_2} + \frac{1}{\sigma}\frac{\partial\sigma}{\partial\xi_2}\right)$$

$$+ \frac{\tilde{h}_3 w\pi}{h_3}\left(-\frac{1}{\xi_3} + \frac{1}{\tau}\frac{\partial\tau}{\partial\xi_3} + \frac{1}{\sigma}\frac{\partial\sigma}{\partial\xi_3}\right) \tag{70}$$

where the inverse relations have been used throughout in the metrics and the (ξ_1, ξ_2, ξ_3) system. The $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are then, respectively, parallel to the σ, τ, π axes.

If the contravariant form for the velocity is assumed, $\overline{v} = \sum_n h_n v^n \hat{e}_n$, where $v^n = v_n/h_n$ and v^n are the components of a contravariant vector, then the transformation of components in R and $\tilde{R}$ is related by

$$\tilde{v}^n = \sum_m v^m \frac{\partial\tilde{\xi}_n}{\partial\xi_m} = \sum_m v^m \left(\frac{h_m}{\tilde{h}_n}\right)^2 \frac{\partial\xi_m}{\partial\tilde{\xi}_n} \tag{71}$$

where $\tilde{\xi}_n$ are τ, σ, π, respectively, for $n = 1, 2, 3$; these forms are essentially Eqs. (68) through (70). In the derivation of the covariant components of Eq. (45), in general,

$$f^i_{,j} = \frac{\partial f^i}{\partial \xi_j} + \sum_m f^m \left\{ {i \atop m\ j} \right\} \qquad \text{(14b) or (72)}$$

where f^i are the covariant components, and $(,j)$ denotes differentiation with respect to ξ_j. In performing operations on ψ, θ derivatives with Eq. (14b), it may be seen that Eq. (45) results.

It remains now to express the original system in terms of the approximate system (for numerical analysis purposes) in terms of (σ, τ, π) coordinates. The form (Eqs. 5a through 9a) is invariant in all coordinate systems herein. From previous considerations, then, the (σ, τ, π) system in $\tilde{R}_1$ is of the general form (excluding the energy equation for the present)

$$\tilde{v}^j \left(\frac{\partial \tilde{v}^i}{\partial x^j} + {}_x\left\{ {i \atop \ell\ j} \right\} \tilde{v}^\ell \right) + \frac{1}{\rho} \tilde{g}^{ij} \frac{\partial p}{\partial x^j} = 0 \qquad \text{(73), (74), (75)}$$

$$(\rho \tilde{v}^i)_{,i} = 0 \qquad \text{(76)}$$

where the metrics of the transformed system are known as combinations of the metrics of the R_1 original system; the velocities of the original system, written in contravariant form are then simply contravariantly transformable into the

$$\frac{\tilde{u}}{\tilde{h}_1} \quad , \quad \frac{\tilde{v}}{\tilde{h}_2} \quad , \quad \frac{\tilde{w}}{\tilde{h}_3}$$

contravariant components in $\tilde{R}_1$.

Here

$$\tilde{h}_1 = \tilde{g}^{11} = K_1^{-1/2} \quad , \quad \tilde{h}_2 = \tilde{g}^{22} = K_2^{-1/2} \quad , \quad \tilde{h}_3 = \tilde{g}^{33} = K_3^{-1/2}$$

In Eqs. (73) through (77) the x^j independent variables represent (σ, τ, π), respectively, for $j = 1, 2, 3$. In that a finite difference approach has been suggested throughout, these transformed equations in $\tilde{R}_1$ with known boundaries are to be solved in $\tilde{R}_1$; solutions in R_1 are then obtained through Eqs. (62) through (70), and (12a). In Eq. (73), p, ρ are taken as <u>scalar</u> quantities and transform as scalars. The new form of the energy equation will now be discussed.

In Ref. 1, $f(\tau y^2) = PR^{-\gamma}$ arises from the integral

$$\frac{d\tau}{(-\tau^2)/y} = dy \qquad (77)$$

On a streamline Eqs. 68 through 70 become

$$\left.\frac{\tilde{u}}{u}\right|_{stm.} = \left.\frac{-\tau\tilde{h}_1}{h_1\xi_1}\right|_{stm.} \tag{78}$$

$$\left.\frac{\tilde{v}}{\tilde{h}_2}\right|_{stm.} = \left.\left[-\frac{u\xi_1}{\left(C^2+\xi_1^2\right)h_1} + \frac{v\xi_2}{h_2\left(1-\xi_2^2\right)}\right]\sigma\right|_{stm.} \tag{79}$$

$$\left.\frac{\tilde{w}}{\tilde{h}_3}\right|_{stm.} = -\pi\left.\left[\frac{\xi_1 u}{\left(C^2+\xi_1^2\right)h_1} + \frac{v}{h_2\xi_2} + \frac{w}{h_3\xi_3}\right]\right|_{stm.} \tag{80}$$

and the u, v, w components in R_1 may be expressed in terms of derivatives of the independent variables by Eq. (27) along a streamline. In general, $\sigma = \sigma(\xi_1, \xi_2, \xi_3; u, v, w, \rho)$, $\tau = \tau(\xi_1, \xi_2, \xi_3; u, v, w, \rho)$, and $\pi = \pi(\xi_1, \xi_2, \xi_3; u, v, w, \rho)$ and inverses were numerically determined in the neighborhood of a point with u, v, w, ρ fixed as $\xi_1 = \xi_1(\sigma, \tau, \pi; u, v, w, \rho)$, $\xi_2 = \xi_2(\sigma, \tau, \pi; u, v, w, \rho)$, $\xi_3 = \xi_3(\sigma, \tau, \pi; u, v, w, \rho)$ or alternatively, as $\xi_1 = \xi_1(\sigma, \tau, \pi; \tilde{u}, \tilde{v}, \tilde{w}, \rho)$, $\xi_2 = \xi_2(\sigma, \tau, \pi; \tilde{u}, \tilde{v}, \tilde{w}, \rho)$, $\xi_3 = \xi_3(\sigma, \tau, \pi; \tilde{u}, \tilde{v}, \tilde{w}, \rho)$. On a streamline, however, ratios of velocities in a particular coordinate frame are known as ratios of derivatives, as indicated for the ϕ, ψ surfaces by Eqs. (40) through (43). Therefore, with Eqs. (28) and (29), the two integrals representing stream surfaces arise from

$$I_1: \left.\frac{-d\tau/\tau}{d\xi_1/\xi_1}\right|_{stm.} = \left.\frac{-d\pi/\pi}{\dfrac{\xi_1 d\xi_1}{C^2+\xi_1^2} + \dfrac{d\xi_2}{\xi_2} + \dfrac{d\xi_3}{\xi_3}}\right|_{stm.} \tag{81}$$

$$I_2: \left. \frac{d\sigma/\sigma}{\dfrac{-\xi_1\, d\xi_1}{C^2 + \xi_1^2} + \dfrac{\xi_2\, d\xi_2}{1 - \xi_2^2}} \right|_{\text{stm.}} \qquad \left. \frac{-d\pi/\pi}{\dfrac{\xi_1\, d\xi_1}{C^2 + \xi_1^2} + \dfrac{d\xi_2}{\xi_2} + \dfrac{d\xi_3}{\xi_3}} \right|_{\text{stm.}} \tag{82}$$

where $\xi_1^2 > C^2$, and $\xi_2^2 < 1$ outside of the $|\epsilon|^2$ stagnation streamtube approximation surface. Provided $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ and the remaining geometrical quantities of Eqs. (78) through (80) are prescribed, then velocity ratios on streamlines are determined. On a streamline, from Eq. (12a), or Eqs. (81) through (82),

$$\tau = \frac{\text{const.}}{\xi_1} \quad , \quad \sigma = \frac{\text{const.}}{\left[\left(C^2 + \xi_1^2\right)\left(1 - \xi_2^2\right)\right]^{1/2}} \qquad (83), (84)$$

$$\pi = \frac{\text{const.}}{\xi_2 \xi_3 \left[\left(C^2 + \xi_1^2\right)\left(1 - \xi_2^2\right)\right]^{1/2}} \tag{85}$$

Next, for a particular σ, τ, π (starting from a location on the shock front), it is then possible to determine relations of the form $\xi_1 = \xi_1(\sigma, \tau, \pi)$, $\xi_2 = \xi_2(\sigma, \tau, \pi)$, $\xi_3 = \xi_3(\sigma, \tau, \pi)$ on streamlines in $\tilde{R}$. Provided the tangent streamline is next constructed over a small distance from the original surface, new (σ, τ, π) values are obtained in such a manner that the streamlines in (σ, τ, π) space, $\tilde{R}$, are a priori determinable. Then in form, locally,

$$p\rho^{-\gamma} = f[I_1(\sigma, \tau, \pi), I_2(\sigma, \tau, \pi)] \tag{86}$$

Although this procedure places heavy emphasis on numerical routines to obtain inverses [and surfaces $u/\tilde{u}$, $v/\tilde{v}$, $w/\tilde{w}$, for $\sigma, \tau, \pi, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ specified from Eqs. (83) through (85), and (78) through (80) such that $\xi_i = \xi_i(\sigma, \tau, \pi; u, v, w) = \xi_i(\sigma, \tau, \pi; \tilde{u}, \tilde{v}, \tilde{w})$, etc.], essentially, these are necessary steps for the particular transformation chosen (Eq. 12).

Also, as $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ = constants, in Eqs. (73) through (75), and for velocities $\tilde{u}, \tilde{v}, \tilde{w}, {}_x\left\{ {i \atop \ell \; j} \right\} = 0$. Provided, next, the Eq. (75) is replaced by Bernoulli's equation, then together with Eq. (86), after elimination of p, ρ terms, three nonlinear coupled equations in terms of $\tilde{u}$, $\tilde{v}, \tilde{w}$ result, similar to the end result of Ref. 1. These equations are then subject to numerical methods for solution in a manner analogous to that of Ref. 1.

V. SOME REMARKS CONCERNING NUMERICAL ANALYSIS SCHEMES FOR IMPROPERLY SET ELLIPTIC SYSTEMS OF EQUATIONS

The system (Eqs. 73 through 76) and the energy equation (Eq. 86) have been implicitly given; the explicit form may be given either in terms of u, v, w or $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ velocity components. In terms of the latter system, and $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ velocities, the system is of the form

$$\begin{pmatrix} C^i_{11} & C^i_{12} & C^i_{13} \\ C^i_{21} & C^i_{22} & C^i_{23} \\ C^i_{31} & C^i_{32} & C^i_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{u}}{\partial x^i} \\ \frac{\partial \tilde{v}}{\partial x^i} \\ \frac{\partial \tilde{w}}{\partial x^i} \end{pmatrix} + \begin{pmatrix} B_{11} \\ B_{22} \\ B_{33} \end{pmatrix} = 0 \qquad (87)$$

where x^i denotes σ, τ, π for i = 1, 2, 3, respectively.

In that the sonic surface may be a priori determined in the R_1 space, the transformation into the $\tilde{R}_1$ space follows. The difference scheme, starting from the shock, is then required to satisfy conditions on the given sonic line. In Eq. (87), the elements of the matrices are, respectively,

$$\rho = \left\{ \left[C^2_\infty - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right] \left(\frac{\gamma - 1}{2\gamma f} \right) \right\}^{1/\gamma-1} \qquad (88)$$

$$p = \left\{ \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right] \left(\frac{\gamma - 1}{2\gamma f} \right) \right\}^{\gamma/\gamma-1} f \tag{89}$$

where $C_\infty^2 = 1 + \dfrac{2}{(\gamma - 1)M_\infty^2}$, and

$$C_{11}^1 = \frac{1}{\tilde{h}_1} \left\{ \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right] \left[\frac{\gamma - 1}{2\gamma f} \right] \right\}^{1/\gamma-1} - 2 \frac{\tilde{u}^2}{\tilde{h}_1} \left(\frac{\gamma - 1}{2\gamma f} \right)^{1/\gamma-1}$$

$$\times \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right]^{-\gamma/\gamma-1} \left(\frac{1}{\gamma - 1} \right)$$

$$C_{11}^2 = - 2 \frac{\tilde{v}\tilde{u}}{\tilde{h}_2} \left(\frac{\gamma - 1}{2\gamma f} \right)^{1/\gamma-1} \left(\frac{1}{\gamma - 1} \right) \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right]^{-\gamma/\gamma-1}$$

$$C_{11}^3 = - \frac{2\tilde{u}\tilde{w}}{\tilde{h}_3} \left(\frac{\gamma - 1}{2\gamma f} \right)^{1/\gamma-1} \left(\frac{1}{\gamma - 1} \right) \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right]^{-\gamma/\gamma-1}$$

$$C_{12}^1 = - 2 \frac{\tilde{u}\tilde{v}}{\tilde{h}_1} \left(\frac{\gamma - 1}{2\gamma f} \right)^{1/\gamma-1} \left(\frac{1}{\gamma - 1} \right) \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right]^{-\gamma/\gamma-1}$$

$$C_{12}^2 = \frac{1}{\tilde{h}_2} \left\{ \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right] \left[\frac{\gamma - 1}{2\gamma f} \right] \right\}^{1/\gamma-1} - 2 \frac{\tilde{v}^2}{\tilde{h}_2} \left(\frac{\gamma - 1}{2\gamma f} \right)^{1/\gamma-1}$$

$$\times \left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \right]^{-\gamma/\gamma-1} \left(\frac{1}{\gamma - 1} \right)$$

$$C_{12}^{3} = -2\,\frac{\tilde{v}\tilde{w}}{\tilde{h}_3}\left(\frac{\gamma-1}{2\gamma f}\right)^{1/\gamma-1}\left(\frac{1}{\gamma-1}\right)\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{-\gamma/\gamma-1}$$

$$C_{13}^{1} = -2\,\frac{\tilde{u}\tilde{w}}{\tilde{h}_1}\left(\frac{\gamma-1}{2\gamma f}\right)^{1/\gamma-1}\left(\frac{1}{\gamma-1}\right)\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{-\gamma/\gamma-1}$$

$$C_{13}^{2} = -2\,\frac{\tilde{v}\tilde{w}}{\tilde{h}_2}\left(\frac{\gamma-1}{2\gamma f}\right)^{1/\gamma-1}\left(\frac{1}{\gamma-1}\right)\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{-\gamma/\gamma-1}$$

$$C_{13}^{3} = \frac{1}{\tilde{h}_3}\left\{\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]\left[\frac{\gamma-1}{2\gamma f}\right]\right\}^{1/\gamma-1} - 2\,\frac{\tilde{w}^2}{\tilde{h}_3}\left(\frac{\gamma-1}{2\gamma f}\right)^{1/\gamma-1}$$

$$\times\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{-\gamma/\gamma-1}\left(\frac{1}{\gamma-1}\right)$$

$$B_{11} = \frac{\tilde{u}}{\tilde{h}_1}\left\{\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma-1}{2\gamma}\right)^{1/\gamma-1}\left(-\frac{f^{-\gamma/\gamma-1}}{\gamma-1}\right)\left(f_{I_1}I_{1_{x^1}} + f_{I_2}I_{2_{x^2}}\right)\right\}$$

$$+\frac{\tilde{v}}{\tilde{h}_2}\left\{\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma-1}{2\gamma}\right)^{1/\gamma-1}\left(\frac{-f^{-\gamma/\gamma-1}}{\gamma-1}\right)\left(f_{I_1}I_{1_{x^2}} + f_{I_2}I_{2_{x^2}}\right)\right\}$$

$$+\frac{\tilde{w}}{\tilde{h}_3}\left\{\left[C_\infty^2 - (\tilde{u}^2+\tilde{v}^2+\tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma-1}{2\gamma}\right)^{1/\gamma-1}\left(\frac{-f^{-\gamma/\gamma-1}}{\gamma-1}\right)\left(f_{I_1}I_{1_{x^3}} + f_{I_2}I_{2_{x^3}}\right)\right\}$$

$$C_{22}^{2} = 0 \qquad C_{22}^{3} = 0 \qquad C_{23}^{2} = 0 \qquad C_{23}^{3} = 0$$

$$C_{21}^{1} = \frac{\tilde{u}}{\tilde{h}_1} - \frac{2\tilde{u}}{\tilde{h}_1}\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}$$

$$\times\left(\frac{\gamma}{\gamma - 1}\right)\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1} f^{-1/\gamma-1}$$

$$C_{21}^{2} = \frac{\tilde{v}}{\tilde{h}_2}$$

$$C_{21}^{3} = \frac{\tilde{w}}{\tilde{h}_3}$$

$$C_{22}^{1} = -\frac{2\tilde{v}}{\tilde{h}_1} f^{-1/\gamma-1}\left(\frac{\gamma}{\gamma - 1}\right)\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

$$\times\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}$$

$$C_{23}^{1} = -\frac{2\tilde{w}}{\tilde{h}_1} f^{-1/\gamma-1}\left(\frac{\gamma}{\gamma - 1}\right)\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

$$\times\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}$$

$$B_{22} = \frac{1}{\tilde{h}_1}\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{\gamma/\gamma-1}\left(- \frac{f^{-\gamma/\gamma-1}}{\gamma - 1}\right)\left(f_{I_1}I_{1_x1} + f_{I_2}I_{2_x1}\right)$$

$$\times\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

$$C_{31}^1 = 0 \qquad C_{31}^3 = 0 \qquad C_{33}^1 = 0 \qquad C_{33}^3 = 0$$

$$C_{31}^2 = - \frac{2\tilde{u}}{\tilde{h}_2} f^{-1/\gamma-1}\left(\frac{\gamma}{\gamma - 1}\right)\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

$$\times\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}$$

$$C_{32}^1 = \frac{\tilde{u}}{\tilde{h}_1}$$

$$C_{32}^2 = \frac{\tilde{v}}{\tilde{h}_2} - \frac{2\tilde{v}}{\tilde{h}_2} f^{-1/\gamma-1}\left(\frac{\gamma}{\gamma - 1}\right)\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

$$\times\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}$$

$$C_{32}^3 = \frac{\tilde{w}}{\tilde{h}_3}$$

$$C_{33}^2 = -\frac{2\tilde{w}}{\tilde{h}_2} f^{-1/\gamma-1}\left(\frac{\gamma}{\gamma - 1}\right)\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

$$\times\left\{\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]\left(\frac{\gamma - 1}{2\gamma f}\right)\right\}^{-1/\gamma-1}$$

$$B_{33} = \frac{1}{\tilde{h}_2}\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\right]^{\gamma/\gamma-1}\left(-\frac{1}{\gamma - 1} f^{-\gamma/\gamma-1}\right)\left(f_{I_1} I_{1_x 2} + f_{I_2} I_{2_x 2}\right)$$

$$\times\left[C_\infty^2 - (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2)\left(\frac{\gamma - 1}{2\gamma f}\right)\right]^{-1/\gamma-1}\left(\frac{\gamma - 1}{2\gamma}\right)^{\gamma/\gamma-1}$$

The system Eq. (87) is then subject to finite difference solution in the manner of Ref. 1; solutions at each point of the mesh may be transformed into the physical R_1 space by the inverse relations between R_1 and $\tilde{R}_1$ previously developed. The above problem is then reduced to simultaneous solution of algebraic equations at points of the mesh.

To conclude, finite difference methods for problems of this nature have been discussed in Refs. 4 and 5 for the plane case of the Laplace equation. A discussion of Ref. 4 has been given in Ref. 1; a brief discussion of Ref. 3 will be given here, in that methods of this type are suggestive of methods to be used for the problem in question.

For the Laplace equation[12] $u_{xx} + u_{yy} = 0$, $u(0, y) = u(\pi, y) = 0$, $u(x, 0) = \phi(x)$, $u_y(x, 0) = \psi(x)$, then $u(x, y) \sqrt{\pi/2} = \sum_{k=1}^{\infty} a_k e^{ky} \sin kx + \sum_{k=1}^{\infty} b_k e^{-ky} \sin kx$; in Čudov's subsequent analysis of the Cauchy problem, the second term of the solution is dropped throughout to simplify the results. Then provided in $0 \leq x \leq \pi$, $0 \leq y \leq Y$, with $\int_0^{\pi} u^2(x, y)\, dx \leq M^2$, the norm on the initial data is taken as

$$||\phi||_0 = \left[\sum_{k=1}^{\infty} a_k^2 e^{2kY} \right]^{1/2} \tag{90}$$

and the norm on u is taken as

$$||u|| = \sup_{0 \leq y \leq Y} \left[\int_0^{\pi} u^2(x, y)\, dx \right]^{1/2} \tag{91}$$

then $||u|| \leq ||\phi||_0$. In terms of these norms, the effect of round-off errors in various difference schemes are determined by assigning approximate values to the function $u(x, 0) = \phi(x)$ on $x = x_n = nh$, with $h = \pi/N$, $n = 1, \cdot \cdot \cdot, N - 1$ and then ascertaining errors between exact and approximate solutions on x_n lines for $0 \leq y \leq Y$. The "size"

[12]Unique solutions that depend continuously on the initial data under certain restrictions on the class of solution functions are considered in Čudov's paper.

of $u(y)$ on the mesh is then taken as

$$||u(y)||_h = \left[h \sum_{n=1}^{N-1} u^2(x_n, y)\right]^{1/2} \tag{92}$$

Next for $u(x, 0) = \phi(x)$ discretized as $u(x_n, 0) = \bar{u}(x_n, 0) - \delta_n$ $n = 1, \ldots, N-1$, then $\bar{u}(x, y) = \sqrt{2/\pi} \sum_{k=1}^{N-1} \bar{a}_k e^{ky} \sin kx$, [where $\bar{a}_k$ are determined from the $\bar{u}(x_n, 0)$ boundary condition] so that the estimate on x_n lines becomes from (Eq. 92),

$$||\bar{u}(y) - u(y)||_h \leq \sqrt{2}\, M \frac{e^{-(N+1)(Y-y)}}{\sqrt{1 - e^{-4N(Y-y)}}} + \delta e^{(N-1)y} \tag{93}$$

where M is a constant and $\delta = \left[h \sum_{n=1}^{N-1} \delta_n^2\right]^{1/2}$. Now for a system of Cauchy-Riemann equations on the uniform rectangular net with steps $\Delta x = h$, $\Delta y = \tau = \tau(h)$ with $\tau(h) \to 0$ as $h \to 0$, a norm on the initial data similar to (Eq. 90) is chosen

$$||\phi||_{0h} = \left[\sum_{k=1}^{N-1} \hat{a}_k^2 e^{2kY}\right]^{1/2} \tag{94}$$

where the $\hat{a}_k$ are coefficients formed from the expansion of $\phi_n = \phi(nh)$ in terms of the orthogonal system $\left\{\sqrt{2/\pi} \sin knh\right\}$, $k = 1, \ldots, N-1$.

Now the difference scheme in τ, h is stable for

$$||u(y)||_h \leq K||\phi||_{0h} \qquad \begin{matrix} 0 \leq y \leq Y \\ 0 < h \leq h_0 \end{matrix} \tag{95}$$

If the difference scheme has solutions of the form $u(mh, n\tau) = u_m^n = s^n(k;h) \sin kmh$, then the necessary condition of stability, from (Eq. 95), is $|s^n(k;h)| \leq Ke^{kY}$, $(0 < h\tau \leq Y)$, with a strong form, $|s^n(k;n)| \leq Ke^{ky}$, $0 \leq n\tau \leq y$. If next $1/\tau [\log s(k;h)] \to k$ as $h \to 0$, for $0 \leq k \leq k_0$, then u_n converges to the exact solution in the norm Eq. (92). Čudov then proceeds to give seven difference schemes with conditions on τ, h such that the strong stability condition is met, and also gives the error of approximation for the Cauchy-Riemann systems, i.e.,

$$\frac{u_m^{n+1} - u_m^n}{\tau} = -\frac{v_{m+1}^n + v_{m-1}^n}{2h}, \quad \frac{v_m^{n+1} - v_m^n}{\tau} = \frac{u_{m+1}^n - u_{m-1}^n}{2h} \tag{96}$$

error of approx. = $e = O(\tau) + O(h^2)$

stable for $\tau/h = C$ = arb. const.,

$$\left.\begin{aligned} \frac{u_m^{n+1} - u_m^n}{\tau} &= - \frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{\tau}{2}\,\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \\ \frac{v_m^{n+1} - v_m^n}{\tau} &= \frac{u_{m+1}^n - u_{m-1}^n}{2h} - \frac{\tau}{2}\,\frac{v_{m+1}^n - 2v_m^n - v_{m-1}^n}{h_2} \end{aligned}\right\} \qquad (97)$$

error of approx. = $e = O(\tau^2) + O(h^2)$

stable for $\tau/h = C$ = arb. const.

etc. Now for the error of approximation comparable to the round-off error (from the stability of solutions with respect to the initial data strong stability criteria), the step size h is determinable so that maximum computational accuracy may be obtained. Analysis of this type is necessary for the problem considered in (σ, τ, π) space; however, extension to nonlinear problems presents considerable additional difficulty so that for purposes of computation, the above analysis serves only as a model for estimation of certain norms such that the computational solution is manageable.

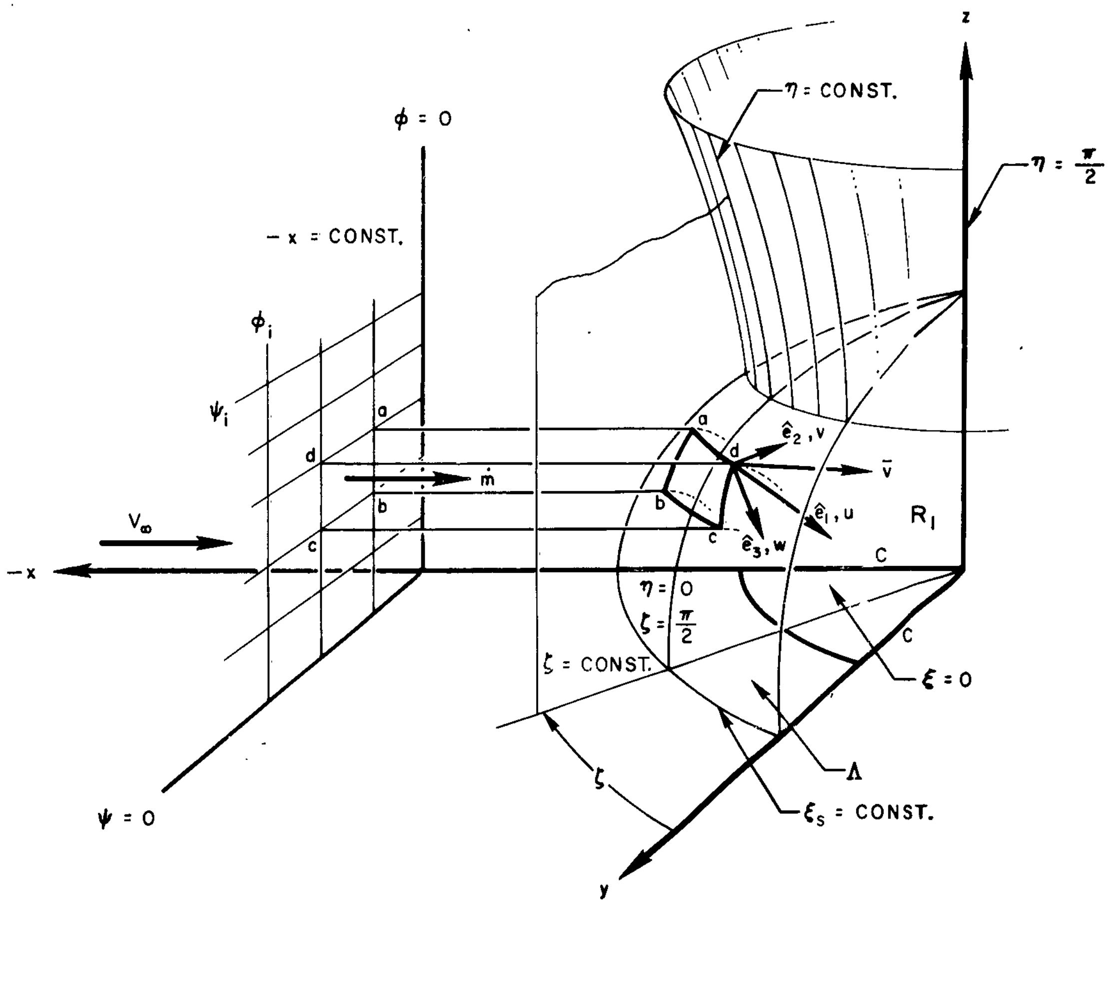

Fig. 1. Coordinate System and Velocity Component System

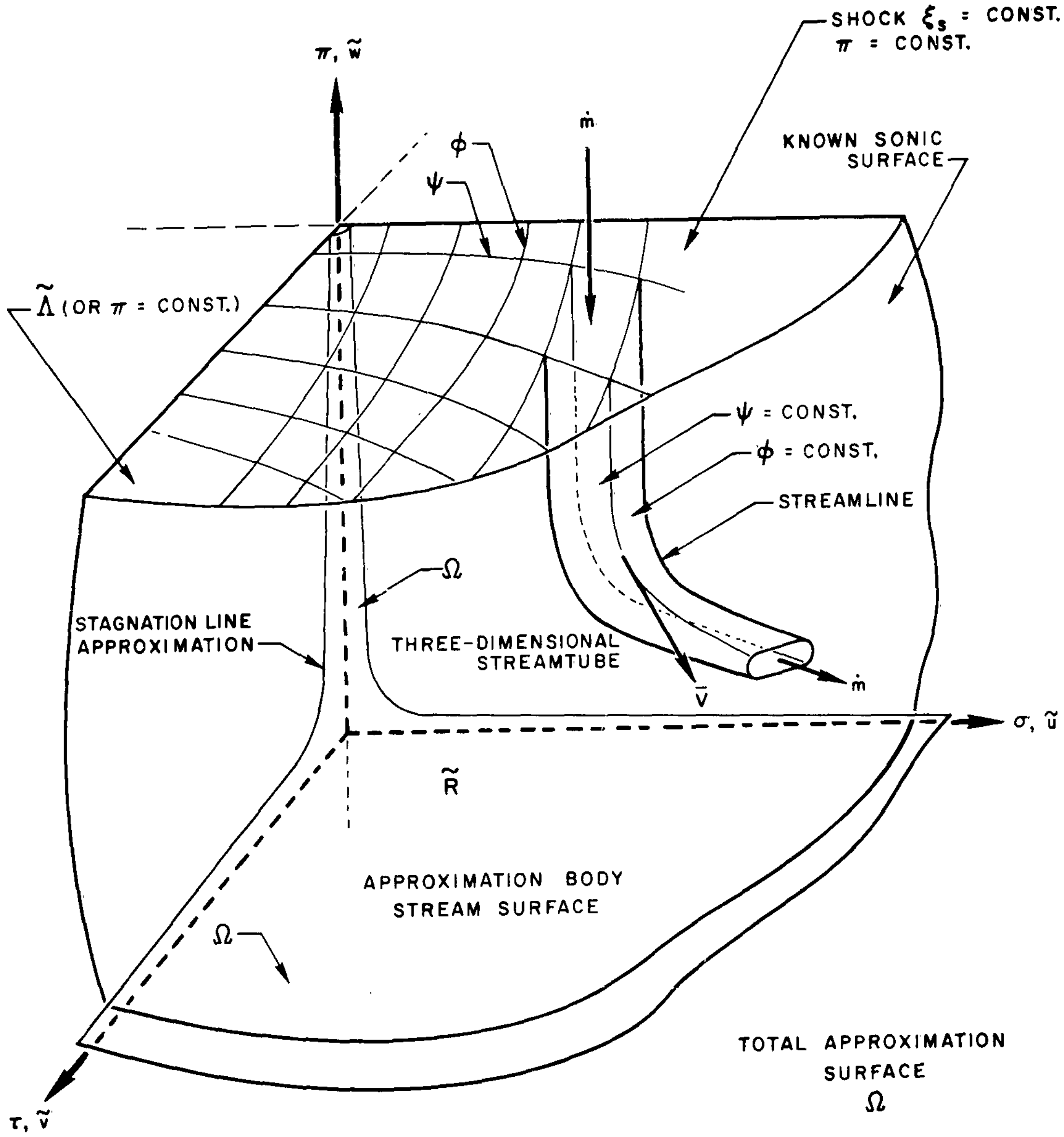

Fig. 2. Transformed Coordinate System

REFERENCES

1. A. Ferri and R. Vaglio-Laurin, "Theoretical Investigation of the Flow Field about Blunt-Nosed Bodies in Supersonic Flight," J. Aeronaut. Sci. 25 (December 1958).

2. R. Vaglio-Laurin, "On the PLK Method and the Supersonic Blunt Body Problem," PIBAL-546, Polytechnic Institute of Brooklyn, New York (1960).

3. R. J. Swigart, "A Theory of Asymmetric Hypersonic Blunt Body Flows," AFOSR-TN-62-2232, SUDAER No. 120, Stanford University, Palo Alto, California (January 1962).

4. C. Ortloff, "A Reflection Method for the Calculation of the Subsonic Flow Field Associated with Small Angle of Attack Blunted Axisymmetric and Asymmetric Bodies for $M_{\infty} >> 1$." Report, Aerospace Corporation, El Segundo, California (to be issued).

5. H. Weinberger, "Error Bounds in Finite-Difference Approximation to Solutions of Symmetric-Hyperbolic Systems," J. Soc. Ind. Appl. Math. (7) 1 (March 1959).

6. F. John, "Advanced Numerical Analysis," New York University Lecture Notes, New York City (1956).

7. P. Lax, "On the Stability of Difference Approximations to Solutions of Hyperbolic Equations with Variable Coefficients," Communs. Pure Appl. Math. (24) (1961).

8. P. Lax, "Difference Approximation to Solutions of Linear Differential Equations," Proceedings of the Conference on Partial Differential Equations, Berkeley (1954).

9. P. Garabedian, "Stability of Cauchy's Problem in Space for Analytic Systems of Arbitrary Type," J. Math. and Mech. (9) 6 (1960).

10. G. Moretti, "Flow Field Analysis of Reentry Configurations by a General Three-Dimensional Method of Characteristics," Supersonic Flow About General Three-Dimensional Blunt Bodies, III, ASD-TR-61-727, Vol. III. General Applied Science Laboratories, Inc., Westbury, Long Island, New York (October 1962).

REFERENCES (Continued)

11. A. Ferri, General Theory of High Speed Aerodynamics, Princeton Series Vol. VI, Sec. G; 24, 25, 26. (Princeton University Press, Princeton, 1954).

12. C. Ortloff, "On Generalized Solutions to the Inviscid Three-Dimensional Perturbation Equations in a Hyperbolic Region, I: Theoretical Results," Report No. TDR-169(3230-11)TN-3, Vol. I. Aerospace Corporation, El Segundo, California (14 September 1962).

13. C. Pucci, "Sui Problemi di Cauchy non <<Ben Posti>>," Rendiconti dell' Accademia Nazionale dei Lincei, Clase di Scienze Fisiche, Matematiche, Naturali, Serie VIII, No. 5 (May 1955).

14. L. A. Čudov, "Difference Methods of Solving the Cauchy Problem for the Laplace Equation," Doklady Akad. Nauk S. S. S. R. (May 1962) pp. 499-503.

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. AN INVERSE METHOD FOR THE DETERMINATION OF THE THREE-DIMENSIONAL SUBSONIC FLOW REGION ABOUT BLUNTED BODIES WITHOUT AXIAL SYMMETRY AT SUPERSONIC FREE STREAM MACH NUMBERS, prepared by C. R. Ortloff. 3 June 1963. [70]p. incl. illus. (Report TDR-169(3230-11)TR-2; SSD-TDR-63-119) (Contract AF 04(695)-169 Unclassified report

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